

Self-similarity, numerical difference, and exact recovery

A central theorem with complete proofs and exact arithmetic certificates

Prepared for Matthew Alan Newman · 13 October 2026

Thesis. OMEGA connects self-similar geometry with recoverable arithmetic. Scaling preserves the marked kernel's proportions. A scalar coordinate gives the same scaling as an affine contraction. Rational approximation and integer quantization introduce distinct, exactly expressible differences. Ordered remainder witnesses make every valid finite integer trajectory recoverable.

Central theorem

Fix the center, orthonormal frame, opening angle and labels of the six-mark kernel. Use the geometric readouts $G(a) = a\phi$ and $C(a) = a\pi$, and let the next scale be the current scale multiplied by $G/(G + C)$. Fix a reference length, a length step $\delta > 0$, known integers $0 < Q < S$, and a finite horizon N . For the canonical Q and S below:

1. The geometric evolution is an exact scaling family, conjugate to the affine map $\Omega(v) = 1 + \lambda(v - 1)$ on $v > 1$.
2. The integer transition is a quantized rational homothety at every nonnegative integer state. It is also a quantized λ -homothety on the proved floor-agreement domain.
3. The total difference at each step and at every finite horizon separates exactly into approximation and quantization contributions.
4. Encoding a valid finite trajectory by its terminal state and ordered remainders is a bijection onto the set of valid witness records.
5. If the initial geometric scale is δd_0 , the same record determines every ideal geometric scale and every affine coordinate as exact expressions.

$$a_j = \frac{\delta \lambda^j}{Q^N} \left(S^N d_N + \sum_{k=0}^{N-1} \rho_k S^k Q^{N-1-k} \right)$$

The proofs below establish all five conclusions. The last formula applies to a validated record; validity includes every intermediate integer inverse.

1 · The marked geometric family

Work in the Euclidean plane with center α and fixed perpendicular unit vectors e_1 and e_2 . Let $0 < \varepsilon < \pi$, $h = \varepsilon/2$, $s = \sin h$ and $c = \cos h$. Thus $s > 0$ and $c > 0$. The canonical opening is $\varepsilon = \pi/648000$. Let $a > 0$ denote the circle radius.

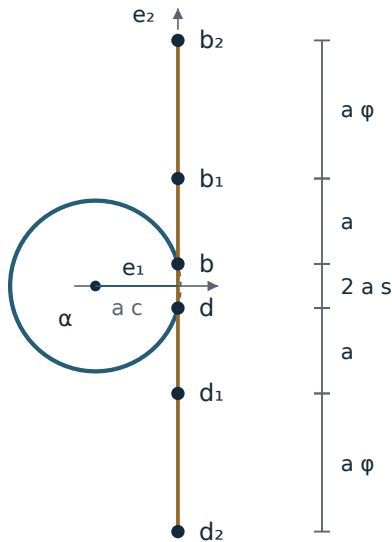
$$\varphi = \frac{1+\sqrt{5}}{2}, \quad A = s + \varphi^2, \quad B = s + 1$$

$$z = (-A, -B, -s, s, B, A)$$

$$p_i(a) = \alpha + a(c e_1 + z_i e_2), \quad i = 1, \dots, 6$$

The labels in order are d_2, d_1, d, b, b_1, b_2 . The seed marks d and b lie on the radius- a circle because $c^2 + s^2 = 1$. Their midpoint is $\alpha + ac e_1$. Their chord length is $2as$. The six scalar z -coordinates are measured from this midpoint.

The kernel $K(a)$ includes these labels, the radius- a circle's retained major arc, and the extension segments. The frame, angular interval and traversal convention remain fixed throughout the scale family. $K(0)$ denotes a formal degenerate boundary object with all marks at α and all lengths zero.



One shape, many scales

Six marks share one affine line.
Their center-relative vectors
span a plane.

G is an outer golden gap.
 C is half the circumference.

The retained major arc has
length $a(2\pi - \varepsilon)$.

Illustration: $\varepsilon = \pi/6$. The canonical opening is $\pi/648000$; no numerical evidence comes from this drawing.

2 · Geometric invariance: proof

Distances and selected readouts

Subtracting the coordinate formulas cancels the center and the common $c e_1$ term. Since e_2 is a unit vector:

$$p_i(a) - p_j(a) = a(z_i - z_j)e_2, \quad \|p_i(a) - p_j(a)\| = a|z_i - z_j|$$

Using $\varphi^2 = \varphi + 1$ gives $A - B = \varphi$ and $B - s = 1$. The ordered adjacent gaps are therefore:

$$a(\varphi, 1, 2s, 1, \varphi)$$

Choose $G(a)$ as either outer golden gap and $C(a)$ as half the seed circle's circumference. These are definite geometric measurements. C is not the retained major-arc length.

$$G(a) = a\varphi, \quad C(a) = a\pi, \quad L_{\text{major}}(a) = a(2\pi - \varepsilon)$$

$$\frac{G(a)}{G(a) + C(a)} = \frac{\varphi}{\varphi + \pi} = \lambda, \quad 0 < \lambda < \frac{1}{2}$$

The bound follows from $\pi > 3$ and $1 < \varphi < 2$; the exact enclosure below also certifies it.

Angles and Gram matrix

The signed bearing of the i -th center-relative vector from e_1 is $\beta_i = \arctan(z_i/c)$. All bearings lie strictly between $-\pi/2$ and $\pi/2$. Their differences have absolute value less than π , so the ordinary center angle equals $|\beta_i - \beta_j|$. The positive bearings are:

$$\beta_4 = h, \quad \beta_5 = \frac{\pi}{4} + \frac{h}{2}, \quad \beta_6 = \arctan\left(\frac{s + \varphi^2}{c}\right)$$

The middle identity follows from $\tan(\pi/4 + h/2) = (1 + \sin h)/\cos h$ and the stated bearing interval. Center-to-mark radii are $a\sqrt{c^2 + z_i^2}$. Let $\mathbf{1}$ denote the six-entry column of ones. Inner products give:

$$\text{Gr}(a) = a^2(c^2\mathbf{1}\mathbf{1}^T + \mathbf{z}\mathbf{z}^T)$$

This matrix has rank at most two. Its first/last principal 2×2 minor is $4a^4c^2A^2 > 0$, so its rank is exactly two. The marks lie on one affine line, while their center-relative vectors span a plane. All normalized distances, bearings and $\text{Gr}(a)/a^2$ are independent of a . This proves the invariance claims.

3 · Geometric and affine compatibility

For $r > 0$ define $H(\alpha, r)(x) = \alpha + r(x - \alpha)$. Applying it to each marked coordinate gives:

$$H_{\alpha, r}(p_i(a)) = \alpha + ra(ce_1 + z_ie_2) = p_i(ra)$$

The same map sends the radius- a circle to the radius- ra circle, preserves its angular interval and orientation, and carries each labeled extension segment to its scaled counterpart.

Hence it carries the complete kernel $K(a)$ to $K(ra)$.

$$H_{\alpha, r}(K_a) = K_{ra}$$

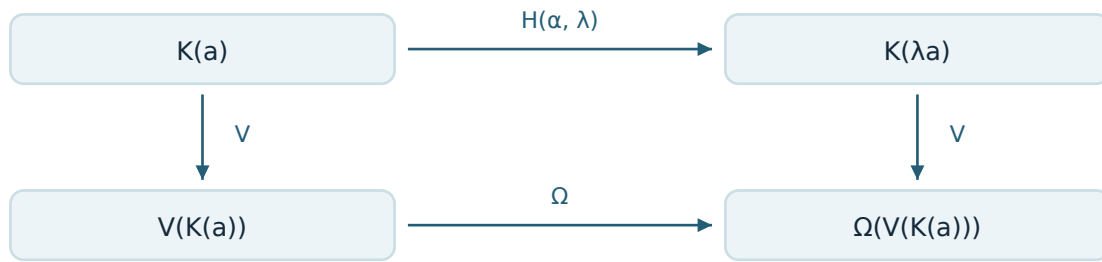
Define the evolution using the measurement ratio $G/(G + C)$. The invariant ratio proves that its update is $H(\alpha, \lambda)$. Induction then gives $K(a_0\lambda^n)$ at stage n .

Exact conjugacy

For a reference length $a_{\text{ref}} > 0$ define $V(K(a)) = 1 + a/a_{\text{ref}}$, and let $\Omega(v) = 1 + \lambda(v - 1)$.

Direct substitution proves:

$$V(H_{\alpha, \lambda}(K_a)) = 1 + \frac{\lambda a}{a_{\text{ref}}} = \Omega(V(K_a))$$



V is a bijection from this positive-scale kernel family onto $v > 1$. Its inverse is $J(v) = K(a_{\text{ref}}(v - 1))$. Thus the commuting identity is an exact conjugacy, and induction gives $\Omega^n(v) = 1 + \lambda^n(v - 1)$. The inverse maps are $H(\alpha, \lambda^{-1})$ and $\Xi(w) = 1 + \lambda^{-1}(w - 1)$; every finite iterate can be inverted.

Every finite scale is positive when $a_0 > 0$. The scales tend to zero, and the affine coordinates tend to 1. In this coordinate, the fixed point 1 represents $K(0)$, the collapsed boundary. Retaining only that limit loses the initial scale.

The evolution rule is part of the mathematical framework. The static shape also admits $H(\alpha, r)$ for every $r > 0$. The theorem proves the compatibility of the chosen measurement-ratio evolution without claiming that shape alone excludes other evolutions.

4 · Approximation and quantization

The canonical arithmetic parameters are:

$$Q = 339949771344778, \quad S = 10^{15}, \quad q = \frac{Q}{S}, \quad \Delta = \lambda - q$$

The exact rational certificate below proves $q < \lambda < (Q + 1)/S$. Thus q is the 15-place decimal truncation of λ and $0 < \Delta < 1/S$. None of the following identities identifies q with λ .

One-step difference theorem

For every nonnegative integer d , Euclidean division of Qd by S gives a unique integer d' and a unique remainder ρ with:

$$d' = \lfloor qd \rfloor, \quad \rho = Qd - Sd', \quad 0 \leq \rho < S$$

Dividing the remainder equation by S gives $qd - d' = \rho/S$. Adding $(\lambda - q)d$ proves the exact decomposition:

$$\lambda d - d' = \Delta d + \frac{\rho}{S}$$

The first term comes from replacing the exact ratio by q . The second comes from projecting qd to its integer part. Both are nonnegative, and the total is strictly positive whenever $d > 0$, because $\Delta > 0$. The Euclidean remainder itself can be zero.

$$\lfloor \lambda d \rfloor - \lfloor qd \rfloor = \lfloor \Delta d + \rho/S \rfloor$$

This follows by taking the floor of $\lambda d = d' + \Delta d + \rho/S$. Thus agreement is equivalent to $\Delta d + \rho/S < 1$; disagreement begins when this total reaches at least one.

Exact connection to geometric quantization

For $\delta > 0$ define $P_\delta(K(a)) = K(\delta \text{ floor}(a/\delta))$, including $K(0)$. For every $d \geq 0$:

$$P_\delta H_{\alpha, q}(K_{\delta d}) = K_{\delta \lfloor qd \rfloor}$$

The proof is substitution: $H(\alpha, q)$ changes the scale to $q\delta d$, and P_δ changes it to $\delta \text{ floor}(qd)$. Where the two floor maps agree, the same statement holds with λ in place of q . Taking $a_{\text{ref}} = \delta$ also gives $\text{floor } \Omega(1 + d) = 1 + d'$ on that domain.

The interval $0 \leq d < 42189065$ is forward invariant under the integer contraction: a positive state strictly decreases. Therefore both quantized floor dynamics have identical complete trajectories when their initial state lies in this interval. This follows by induction from the one-step agreement proved below.

5 · Accumulated numerical difference

Let $d_0, \dots, d_N$ be the rational floor trajectory. Put $\eta_k = \rho_k/S$. Its recurrence is $d_{k+1} = qd_k - \eta_k$. The exact ideal scale in units of δ is $g_N = \lambda^N d_0$.

The accumulated difference theorem

For every $N \geq 0$, with empty sums interpreted as zero:

$$d_N = q^N d_0 - \sum_{k=0}^{N-1} q^{N-1-k} \eta_k$$

$$\lambda^N d_0 - d_N = (\lambda^N - q^N) d_0 + \sum_{k=0}^{N-1} q^{N-1-k} \eta_k$$

$$\lambda^N d_0 - d_N = \sum_{k=0}^{N-1} \lambda^{N-1-k} (\Delta d_k + \eta_k)$$

Proof. The first identity is true at $N = 0$. Multiplying its N -th instance by q and subtracting η_N proves the $N + 1$ instance. Subtracting it from $\lambda^N d_0$ gives the second identity. For the third, write $d_{k+1} = \lambda d_k - (\Delta d_k + \eta_k)$ and repeat the same induction with λ . Thus both descriptions give the same accumulated difference, with their different propagation weights fully accounted for.

Explicit finite bounds

For $N \geq 1$, each $0 \leq \eta_k < 1$. Summing a finite geometric series gives:

$$0 \leq q^N d_0 - d_N < \frac{1 - q^N}{1 - q}$$

The elementary factorization $x^N - y^N = (x - y) \sum_{j=0}^{N-1} x^{N-1-j} y^j$ gives:

$$\lambda^N - q^N = \Delta \sum_{j=0}^{N-1} \lambda^{N-1-j} q^j$$

Since $0 < q < \lambda$, every summand is at most λ^{N-1} . Consequently:

$$0 \leq \lambda^N d_0 - d_N < N \Delta \lambda^{N-1} d_0 + \frac{1 - q^N}{1 - q}$$

These identities hold at every integer seed, including seeds beyond the floor-agreement interval. The real length difference is δ times the stated difference. The finite upper bound is useful but is not an assertion that the actual error approaches that bound.

The general identities are deductive consequences of the recurrence. The verifier also checks their polynomial coefficients with the exact ratio represented by an indeterminate X , so these checks use no floating approximation to λ .

6 · The exact witness record

Fix known Q , S and N . A valid trajectory is an integer sequence $d_0, \dots, d_N$ with $d_k \geq 0$ and $d_{k+1} = \text{floor}(Qd_k/S)$. Its ordered witness record is $(d_N; \rho_0, \dots, \rho_{N-1})$.

Local inverse

$$Qd_k = Sd_{k+1} + \rho_k, \quad d_k = \frac{Sd_{k+1} + \rho_k}{Q}$$

For a genuine trajectory the numerator is divisible by Q . Given a candidate record, start at its terminal state and evaluate this expression for $k = N - 1$ down to 0. Require a nonnegative integer result at every step and require every remainder to be an integer in $[0, S)$.

Validity and bijection theorem

Claim. These checks accept exactly the valid witness records, and an accepted record has exactly one reconstructed trajectory.

Proof of sufficiency. Suppose all checks pass. At step k the reconstructed integer d_k obeys $Qd_k = Sd_{k+1} + \rho_k$ with $0 \leq \rho_k < S$. Divide by S :

$$d_{k+1} \leq \frac{Qd_k}{S} < d_{k+1} + 1$$

Hence $\text{floor}(Qd_k/S) = d_{k+1}$. This holds at every reconstructed step, so the record comes from a valid trajectory. Each inverse expression has one value, so no other trajectory can produce that record.

Proof of necessity. Every valid trajectory obeys Euclidean division, so its remainder bounds, integer inverses and nonnegativity checks all pass. Encoding the reconstructed trajectory reproduces the supplied record. Decoding the encoding of a valid trajectory returns that trajectory.

$$\{\text{valid finite trajectories}\} \longleftrightarrow \{\text{valid witness records}\}$$

At $N = 0$ the record has no remainders and the trajectory consists only of $d_0 = d_N$. The theorem remains true. The proof establishes recoverability as injectivity of the witness encoding; the endpoint projection by itself is generally not injective.

Arithmetic validity establishes a consistent encoded history. Distinct valid records may represent distinct histories with the same endpoint.

7 · Global identity and local validity

The telescoping reconstruction formula

Multiply $Qd_k - Sd_{k+1} = \rho_k$ by $S^k Q^{N-1-k}$ and sum from $k = 0$ to $N - 1$. The contribution involving each interior d_k cancels between its two neighboring terms. Only $Q^N d_0$ and $-S^N d_N$ remain:

$$Q^N d_0 = S^N d_N + \sum_{k=0}^{N-1} \rho_k S^k Q^{N-1-k}$$

Dividing by Q^N gives the initial integer state directly for any valid record. The backward inverse supplies all intermediate states.

A global identity alone does not establish validity

This distinction matters because Q and S need not be coprime. For the canonical parameters, take $N = 2$ and the candidate record:

$$d_2 = 0, \quad \rho_0 = 2Q - \frac{S}{2} = 179899542689556$$

$$\rho_1 = \frac{Q}{2} = 169974885672389$$

Both remainders lie in $[0, S)$. Its global numerator is:

$$Q\rho_0 + S\rho_1 = 2Q^2$$

Thus the summed identity returns the nonnegative integer $d_0 = 2$. Nevertheless, the very first backward step gives:

$$d_1 = \frac{Sd_2 + \rho_1}{Q} = \frac{1}{2}$$

The candidate record is invalid. This explicit counterexample proves that global divisibility and remainder bounds do not, for these parameters, replace the stepwise inverse checks.

The combined theorem therefore has two parts: the telescoping identity proves the global relation, and the local validity theorem proves that a supplied record represents an actual integer orbit. The provided decoder checks both the integer inverse and the reconstructed forward quotient at every step.

8 · What information the witnesses preserve

Quantization fibers

An integer d maps to an endpoint t precisely when $S_t \leq Qd < S(t + 1)$. Solving these inequalities for integer d gives the exact fiber:

$$T_q^{-1}(t) = \left\{ \left\lceil \frac{St}{Q} \right\rceil, \dots, \left\lceil \frac{S(t+1)}{Q} \right\rceil - 1 \right\}$$

Its size is the difference of the two ceiling expressions. Canonically, $2 < S/Q < 3$, so every fiber has two or three elements. For example, 0, 1 and 2 all map to 0. This proves the endpoint alone loses predecessor information.

If two predecessors d and e have the same endpoint, their remainders differ by $Q(d - e)$. Distinct predecessors therefore have distinct witnesses. This is the local reason the augmented transition (endpoint, remainder) is injective.

A smaller sufficient witness on a bounded domain

The remainder is $\rho(d) = Qd \text{ modulo } S$. Let $g = \gcd(Q, S)$ and $M = S/g$. Then:

$$\rho(d) = \rho(e) \iff M \mid (d - e)$$

Proof. Equality of the remainders is equivalent to S dividing $Q(d - e)$. Dividing by g gives M dividing $(Q/g)(d - e)$. Because Q/g and M are coprime, this is equivalent to M dividing $d - e$.

Therefore the first remainder is injective on every interval of consecutive integers of length at most M . Canonically $g = 2$ and $M = 5 \times 10^{14}$. The modular inverse is:

$$(Q/2)^{-1} \equiv 281419755786509 \pmod{5000000000000000}$$

$$d_0 \equiv 281419755786509 (\rho_0/2) \pmod{5000000000000000}$$

For $0 \leq d_0 < M$, this congruence identifies d_0 uniquely, so it identifies the entire deterministic future. It includes the canonical 15-bit domain. Thus the complete ordered ledger is sufficient for general reconstruction, while some bounded domains admit a smaller record.

Why order is part of the general record

For $Q = 1$, $S = 2$ and terminal state 0, the ordered remainders (1,0) decode to (1,0,0), whereas (0,1) decode to (2,1,0). Both records are valid. Their unordered remainders coincide, but their histories differ. This supplies a direct example of why a general decoder needs the ordered sequence.

9 · First disagreement: a denominator proof

Define the rational enclosure endpoints:

$$\ell = \frac{339949771344778929576702809321}{10^{30}}, \quad u = \ell + 10^{-30}$$

The next section proves $\ell < \lambda < u$ using elementary exact bounds. Exact integer cross multiplication gives the following ordering:

$$\frac{846547}{2490212} < q < \frac{14342163}{42189065} < \ell < \lambda < u < \frac{13495616}{39698853}$$

Write the left and right endpoint fractions as p_L/r_L and p_R/r_R . Their integer determinant and denominator sum are:

$$p_R r_L - p_L r_R = 1, \quad r_L + r_R = 42189065$$

Denominator obstruction

If an integer fraction m/d lies strictly between these fractions, with $d > 0$, the following are positive integers:

$$r_L m - p_L d \geq 1, \quad p_R d - r_R m \geq 1$$

Their determinant-one identity gives:

$$d = r_R(r_L m - p_L d) + r_L(p_R d - r_R m) \geq r_R + r_L$$

This proof does not assume that m/d is reduced.

Application to the floor maps

Because $q < \lambda$, any disagreement at $d > 0$ supplies an integer m with $qd < m \leq \lambda d$: choose $m = \text{floor}(qd) + 1$. The fraction m/d lies strictly between the two endpoint fractions. Thus a disagreement requires $d \geq 42189065$.

At $d^* = 42189065$, exact rational evaluation proves:

$$\lfloor qd^* \rfloor = 14342162, \quad 14342163 < \ell d^* < u d^* < 14342164$$

The enclosure then gives $\text{floor}(\lambda d^*) = 14342163$. At $d = 0$ both floors are zero. Consequently:

$$\lfloor qd \rfloor = \lfloor \lambda d \rfloor \quad (0 \leq d < 42189065, \, d \text{ integer})$$

The integer 42189065 is the first nonnegative disagreement. On the positive agreement domain, the one-step difference $\Delta d + \rho/S$ is exactly the fractional part of λd and lies strictly between zero and one.

At the first disagreement, that same total lies strictly between one and two. The approximation and quantization differences therefore explain the floor change through the exact criterion established in Section 4.

10 · Certified enclosure: elementary proof

Alternating sums with explicit remainders

For $0 < x \leq 1$, integrate the finite geometric-series identity for $1/(1+t^2)$:

$$\arctan x = A_n(x) + (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} dt$$

$$A_n(x) = \sum_{j=0}^{n-1} \frac{(-1)^j x^{2j+1}}{2j+1}$$

The integral is strictly positive. Hence $A_{40}(x) < \arctan x < A_{41}(x)$. Every partial sum at $x = 1/5$ or $1/239$ is rational.

Machin's identity and its branch

Put $A = \arctan(1/5)$ and $B = \arctan(1/239)$. The double-angle formulas give $\tan(2A) = 5/12$ and $\tan(4A) = 120/119$. Subtraction then gives $\tan(4A - B) = 1$ exactly. Also $A > 1/5 - 1/375$ and $B < 1/239$, so $4A - B > 0$. The upper bound $A < 1/5$ gives $4A - B < 4/5 < 1$.

Since $\arctan(1) = \int_0^1 1/(1+t^2) dt > 1/2$, $\pi/2 > 1$. Thus $4A - B$ lies in $(0, \pi/2)$, where tangent is injective. It equals $\pi/4$. Therefore $\pi = 16A - 4B$.

$$\pi_- = 16A_{40}(1/5) - 4A_{41}(1/239)$$

$$\pi_+ = 16A_{41}(1/5) - 4A_{40}(1/239), \quad \pi_- < \pi < \pi_+$$

Square-root and ratio bounds

Set $M = 10^{60}$ and $t = \text{floor}\sqrt{5M^2}$. Its exact integer value is:

2236067977499789696409173668731276235440618359611525724270897

Squaring verifies $t^2 < 5M^2 < (t+1)^2$. Define $\varphi_- = (1 + t/M)/2$ and $\varphi_+ = (1 + (t+1)/M)/2$. The function $x/(x+y)$ increases with positive x and decreases with positive y , so:

$$L = \frac{\varphi_-}{\varphi_- + \pi_+} < \lambda < \frac{\varphi_+}{\varphi_+ + \pi_-} = U$$

Exact rational comparison proves $\text{ell} < L < U < u$. This comparison is a finite integer cross-multiplication using the displayed sums and t . The accompanying standard-library verifier evaluates all those rational expressions exactly and also checks $q < \text{ell}$ and $u < (Q+1)/S$. No floating-point value enters this certificate.

11 · Recovering the ideal geometry

Let $a_0 = \delta d_0$ with a known length step $\delta > 0$. The ideal unquantized OMEGA evolution has scale $a_j = \delta \lambda^j d_0$. Let a valid witness record encode an N-step rational integer trajectory from this same d_0 .

Exact geometric recovery corollary

Substitute the global reconstruction identity into the ideal scale formula:

$$a_j = \frac{\delta \lambda^j}{Q^N} \left(S^N d_N + \sum_{k=0}^{N-1} \rho_k S^k Q^{N-1-k} \right), \quad j \geq 0$$

This determines every ideal scale as an exact mathematical expression. Substitution into $p_i(a_j)$ recovers all six marks, the retained arc and the extensions. Substitution into V recovers every affine coordinate. The center, frame, opening, labels, δ and a_{ref} are required known geometric data.

$$p_i(a_j) = \alpha + a_j(c e_1 + z_i e_2), \quad v_j = 1 + \frac{a_j}{a_{\text{ref}}}$$

Proof. Validity uniquely determines d_0 . The geometric evolution and affine coordinate are deterministic functions of that seed and the known geometric parameters. The displayed expression is exactly their substitution. It applies for every finite j , even when the recorded integer trajectory has already reached zero.

This gives a precise bridge between recoverable integer information and ideal self-similarity. Numerical approximation need not equal the exact ratio for the encoded seed to determine the exact symbolic geometry.

The initial quantization boundary

For a general real initial scale a_0 , encoding only $d_0 = \text{floor}(a_0/\delta)$ identifies the interval $\delta d_0 \leq a_0 < \delta(d_0 + 1)$. It does not identify a single real scale. If the initial fractional witness $\theta_0 = a_0/\delta - d_0$ is retained, then:

$$a_j = \delta \lambda^j (d_0 + \theta_0), \quad 0 \leq \theta_0 < 1$$

Thus exact recovery of an arbitrary real initial scale requires its initial fractional data as well. The integer ledger alone recovers the integer history and, for a grid-encoded initial scale, the corresponding exact geometric history.

The proof treats λ symbolically. Producing a finite decimal display of a_j is an additional numerical representation, with its own approximation.

12 · Indefinite scaling and finite extinction

The two stage sequences

For $a_0 > 0$ and $0 < \lambda < 1$, $a_0 \lambda^n > 0$ for every finite n . No finite ideal stage collapses; the limit is zero. Every finite ideal stage remains invertible.

For a positive integer state, $0 < q < 1$ gives $\text{floor}(qd) \leq d - 1$. Thus the integer trajectory reaches zero in finite time. The sharper inductive estimate is:

$$0 \leq d_n \leq q^n d_0$$

If $q^n d_0 < 1$, the nonnegative integer d_n must equal zero. For every canonical seed $0 \leq d_0 \leq 32767$, the exact integer inequality $Q^{10} \times 32767 < S^{10}$ proves extinction by step 10. Zero remains zero and all later remainders are zero. The witness record can therefore be padded to the canonical horizon $N = 23$.

Although every canonical terminal state at horizon 23 is zero, the valid ordered records remain distinct for the 32768 initial seeds. Exact reconstruction follows from the inverse theorem. The endpoint has discarded those distinctions; the record retains them.

Repeated quantization is a distinct operation

Projection after every step generally differs from projection only after an ideal multi-step evolution. A canonical exact example is:

$$26 \mapsto 8 \mapsto 2, \quad [26\lambda^2] = 3$$

The integer path is evaluated using Q/S . Its two inputs are inside the floor-agreement domain, so λ gives the same stepwise path. Meanwhile $3 < 26\ell^2 < 26u^2 < 4$ certifies the single-round ideal floor as 3. This proves the distinction.

The resulting central claim

OMEGA's self-similarity, affine coordinate, numerical differences and reconstructible integer history form one explicit mathematical framework. Exact conjugacy connects the ideal descriptions. Exact error identities measure the numerical changes. Witness encoding makes the quantized finite trajectory injectively representable and recoverable.

The theorem's hypotheses identify the evolution rule and geometric parameters, the arithmetic parameters, the quantizer and the valid record. Under these hypotheses every asserted connection has been proved. The conclusions concern the mathematical relationships and the information actually encoded by the record.

13 · Execution evidence and reproducibility

The accompanying file `omega_geometry_recovery_verify.py` uses Python's standard library only. All acceptance checks use explicit exceptions. It contains no floating-point calculations and no assert statements.

Verified executions

Both normal execution and optimized execution exited with status 0. Their output was byte-for-byte identical. Each reported all eight groups as passing.

1 · Symbolic geometry. Exact coefficient identities in $Q(\varphi)[s]$ for gaps, seed radii and the Gram minor.

2 · Ratio enclosure. Exact arctangent sums, Machin tangent identities, integer-square-root bounds and rational comparisons.

3 · First disagreement. Exact fraction ordering, determinant, denominator sum and threshold floors.

4 · Accumulated difference. 55 seed/horizon cases; identities checked as polynomials in an indeterminate X .

5 · Record validity. Malformed records rejected, including the canonical global-identity counterexample.

6 · Information and extinction. Fiber intervals, modular inverse, ordering example and the exact step-10 bound.

7 · Canonical domain. 32768 states, 753664 transitions/prefix invariants, 32768 complete round trips and first-residue inversions.

8 · Optimization robustness. AST inspection found zero assert statements.

Run the verifier

```
python3 omega_geometry_recovery_verify.py
python3 -O omega_geometry_recovery_verify.py
```

The general geometry and reconstruction statements are proved in this document. The executable checks give reproducible exact algebra and arithmetic evidence for their declared scope. They do not substitute finite sampling for the general proofs. No proof-assistant certificate was produced.

Source integrity

Verifier SHA-256:

```
e044133aceefb442bfa226bdf61359a5
8bca413e12196f090ff19eef3ef84ab5
```

Interpreter used for this record: 3.12.14 (main, Aug 25 2026, 14:00:49) [Clang 22.1.3]

The complete normal and optimized logs were compared during preparation. The proof and its rational certificates are self-contained: the arctangent, square-root, conjugacy, denominator and reconstruction arguments are all stated above.