

Exact Positive Icosahedral Cubature on Fixed Radial Class-I Grids

Quadratic-field interoperability, exact replay,
and typed compatibility with Papers I–V

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Status and dependency boundary. The cubature theorems are consequences of the fixed radial Class-I geometry, full icosahedral symmetry, and exact invariant moment systems over $\mathbb{Q}(\sqrt{5})$. The OMEGA quotient algebra is used only to encode coefficients already obtained from that geometry. Its 15-bit floor contract is a separate finite replay result. The Papers I–V global object supplies a typed ambient assembly and an exact coordinate-transport interface; it is not an antecedent of the cubature proof. The all-frequency statement remains a conjecture from frequency ten onward. No claim of novelty, priority, peer review, or journal acceptance is made.

Abstract

Let $X_f \subset \mathbb{S}^2$ be the complete frequency- f radial Class-I icosahedral grid. Using the full reflection group H_3 , Sobolev’s invariant-cubature reduction, and exact arithmetic in $\mathbb{Q}(\sqrt{5})$, we certify strictly positive orbit-constant probability rules on X_f for $1 \leq f \leq 9$. Their exact algebraic degrees are

$$5, 9, 11, 15, 17, 21, 23, 27, 29,$$

respectively. Exact nonzero residuals at the first omitted invariant degrees prove that these are exact degrees, not only lower bounds.

For $f \leq 8$ the invariant systems are square and give unique rules. At $f = 9$, the exact 11-by-12 system has rank 11 and a one-dimensional affine solution space $W(t)$. Its complete strict-positivity interval is certified exactly in $\mathbb{Q}(\sqrt{5})$, and the exact specialization $W_{333} = t = 1/27$ gives a convenient rational witness. We also determine the exact member maximizing the minimum total orbit mass and, separately, the exact member maximizing the minimum per-node weight. No floating-point feasibility or sign decision is used in these proofs.

Independently, the quotient algebra

$$K_\Omega = \mathbb{Q}(T)[\lambda]/(\lambda^2 - T\lambda - T^2)$$

admits the exact identification $\sqrt{5} = 2\lambda/T - 1$. Hence every cubature coefficient $a + b\sqrt{5}$ has the two-channel representation $(a - b) + (2b/T)\lambda$. This is a genuine coefficient-field interoperability map, but it is not a source of the icosahedral theorem.

Finally, conditional on the frozen strict enclosure of the distinguished real $\lambda_\Omega = \varphi/(\varphi + \pi)$, exact integer replay verifies

$$\lfloor \lambda_\Omega d \rfloor = \left\lfloor \frac{339949771344778}{10^{15}} d \right\rfloor \quad (0 \leq d < 2^{15}).$$

A determinant-one Farey certificate proves that the first disagreement over all nonnegative integers occurs at $d = 42,189,065$. All large coefficients and proof decisions are preserved in the deterministic verification record identified in Section 10.

The global-object integration extends the multi-sorted object associated with Papers I–V by a cubature sort. Its radial branch already contains the equal-weight measure μ_f on X_f ; the present weights define a different positive, full-support measure ν_f on the same nodes. For $f \geq 2$, $\nu_f(H_6) = 0$, so orbit reweighting cancels the equal-weight degree-six obstruction rather than reinterpreting it. Exact SRSD encoding transports the rule without changing any decoded moment. This is a conservative typed interface, not a new derivation of either branch.

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1 Scope and separation of claims

This paper joins several exact statements at typed interfaces, not at a common mathematical origin:

Layer	Statement	Status
Radial geometry	Definition and symmetry of X_f , with $ X_f = 10f^2 + 2$	Imported from [5]
Invariant reduction	H_3 -invariant exactness reduces to invariant moments	Classical; hypotheses checked [8]
Positive cubature	Exact positive rules and first failures for $1 \leq f \leq 9$	Exact finite computation here
OMEGA algebra	$\mathbb{Q}(\sqrt{5})$ embeds by $\sqrt{5} \mapsto 2\lambda/T - 1$	Exact algebra here
Finite floor bridge	15-bit equality and first-disagreement minimality	Exact, conditional on the frozen enclosure
Papers I–V object	Adjoin the cubature measures and exact interface laws	Conservative typed extension of [7]
All frequencies	Positive rules through the proposed threshold for $f \geq 10$	Open conjecture

Deleting the OMEGA sections and files leaves every cubature theorem and proof intact. Conversely, the OMEGA algebra acquires no geometric theorem merely by representing a coefficient produced by the cubature calculation. Likewise, deleting the SRSD coordinate-transport interface leaves the native cubature calculation unchanged.

2 Fixed Class-I grids and orbit masses

Put

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

and take the icosahedral face with vertices

$$v_1 = (0, 1, \varphi), \quad v_2 = (1, \varphi, 0), \quad v_3 = (\varphi, 0, 1).$$

For integers $a, b, c \geq 0$ with $a + b + c = f$, define

$$p_{abc} = av_1 + bv_2 + cv_3, \quad x_{abc} = \frac{p_{abc}}{\|p_{abc}\|}.$$

The complete grid X_f is the full icosahedral orbit union of these barycentric nodes, with duplicates on common edges and vertices identified. The exact count

$$|X_f| = 10f^2 + 2$$

and the radial construction are imported from [5].

Let H_3 denote the full icosahedral reflection group. Its orbits on X_f have representatives

$$\Lambda_f = \{(a, b, c) : a \geq b \geq c \geq 0, a + b + c = f\}.$$

If $\mathcal{O}_{abc} = H_3 x_{abc}$, its size is 12 at a vertex, 30 at an edge midpoint, 60 at a nonmidpoint edge node, and, for an interior face node, 20, 60, or 120 according as all three barycentric entries are equal, exactly two are equal, or all are distinct. These sizes sum to $10f^2 + 2$.

Definition 2.1 (Orbit-mass rule). For total orbit masses W_{abc} , define

$$Q_f[g] = \sum_{(a,b,c) \in \Lambda_f} \frac{W_{abc}}{|\mathcal{O}_{abc}|} \sum_{x \in \mathcal{O}_{abc}} g(x).$$

Thus W_{abc} , not $W_{abc}/|\mathcal{O}_{abc}|$, is the total probability assigned to the orbit. The rule is strictly positive when every $W_{abc} > 0$, and it is probability normalized when $\sum_{\Lambda_f} W_{abc} = 1$.

Throughout, σ denotes normalized spherical area: $\int_{\mathbb{S}^2} 1 \, d\sigma = 1$.

3 Invariant reduction and exact moments

Let

$$V_{\text{ico}} = \{(0, \pm 1, \pm \varphi), (\pm 1, \pm \varphi, 0), (\pm \varphi, 0, \pm 1)\}, \quad R^2 = \varphi + 2,$$

and write $\hat{v} = v/R$. With P_n the Legendre polynomial of degree n , define on $\mathbb{S}^2$

$$H_n(x) = \sum_{v \in V_{\text{ico}}} P_n(x \cdot \hat{v}), \quad n = 6, 10.$$

The normalizations used by the certificate are

$$P_6(t) = \frac{231t^6 - 315t^4 + 105t^2 - 5}{16}$$

and

$$P_{10}(t) = \frac{46189t^{10} - 109395t^8 + 90090t^6 - 30030t^4 + 3465t^2 - 63}{256}.$$

Their homogeneous extensions are H_3 -invariant harmonic polynomials. In particular,

$$\begin{aligned} H_6(V) &= \frac{132}{25}, & H_6(E) &= -\frac{33}{20}, \\ H_{10}(V) &= \frac{988}{625}, & H_{10}(E) &= -\frac{247}{160}, \end{aligned}$$

on a fivefold vertex axis V and a twofold edge-midpoint axis E . Also

$$\int_{\mathbb{S}^2} H_6 \, d\sigma = \int_{\mathbb{S}^2} H_{10} \, d\sigma = \int_{\mathbb{S}^2} H_6 H_{10} \, d\sigma = 0, \quad \int_{\mathbb{S}^2} H_6^2 \, d\sigma = \frac{1584}{325}.$$

Lemma 3.1 (Invariant test set). *Let Q be an H_3 -invariant rule on $\mathbb{S}^2$. Then Q is exact for every spherical polynomial of degree at most t if and only if*

$$Q[H_6^r H_{10}^s] = \int_{\mathbb{S}^2} H_6^r H_{10}^s \, d\sigma \quad \text{whenever} \quad 6r + 10s \leq t.$$

Proof. The invariant ring of the reflection group H_3 has homogeneous basic degrees 2, 6, 10. On $\mathbb{S}^2$, the degree-two invariant is the constant 1. Replacing the degree-six and degree-ten generators by their harmonic parts is a triangular change of generators, so the restricted invariant algebra is generated by H_6 and H_{10} . Sobolev's invariant-cubature criterion then says that an H_3 -invariant rule is exact through degree t if and only if it is exact on all invariant polynomials through degree t [8]. The displayed monomials are a basis for those restricted invariants. $\square$

Every spherical moment used below is obtained from the exact monomial identity

$$\int_{\mathbb{S}^2} x^{2i} y^{2j} z^{2k} \, d\sigma = \frac{(2i-1)!!(2j-1)!!(2k-1)!!}{(2i+2j+2k+1)!!},$$

with $(-1)!! = 1$; a monomial with an odd exponent has mean zero. Consequently all moment matrices and right-hand sides lie in $\mathbb{Q}(\sqrt{5})$.

4 Exact positive rules for frequencies one through nine

Define

$$t_f = \begin{cases} 3f + 2, & f \text{ odd}, \\ 3f + 3, & f \text{ even}. \end{cases}$$

Theorem 4.1 (Certified fixed-grid cubature). *For every $1 \leq f \leq 9$, the fixed grid X_f supports a strictly positive H_3 -orbit-constant probability rule of exact algebraic degree t_f . For $1 \leq f \leq 8$, the rule satisfying the required invariant moments is unique. For $f = 9$, the required moment equations have a one-dimensional affine solution space; one exact positive member is selected by*

$$W_{333} = \frac{1}{27}.$$

f	$ X_f $	orbits	t_f	first failure	smallest W , approx.
1	12	1	5	6	1.000000
2	42	2	9	10	0.238095
3	92	3	11	12	0.080984
4	162	4	15	16	0.053869
5	252	5	17	18	0.030520
6	362	7	21	22	0.021749
7	492	8	23	24	0.015328
8	642	10	27	28	0.012087
9	812	12	29	30	0.006601

Proof. For $\alpha = (a, b, c) \in \Lambda_f$, form the invariant moment vector

$$v_\alpha = (H_6(x_\alpha)^r H_{10}(x_\alpha)^s)_{6r+10s \leq t_f}$$

and the spherical target

$$b_f = \left(\int_{\mathbb{S}^2} H_6^r H_{10}^s d\sigma \right)_{6r+10s \leq t_f}.$$

Let M_f have columns v_α . For $1 \leq f \leq 8$, the number of rows equals $|\Lambda_f|$. Exact inversion over $\mathbb{Q}(\sqrt{5})$ gives

$$M_f W_f = b_f.$$

The exact verifier constructs this inverse, reduces every entry to $a + b\sqrt{5}$ with $a, b \in \mathbb{Q}$, checks all residuals exactly, and proves every coordinate of W_f positive by rational sign comparisons. Lemma 3.1 gives exactness through t_f .

At the least invariant degree above t_f , the verifier evaluates, respectively for $f = 1, \dots, 8$,

$$H_6, H_{10}, H_6^2, H_6 H_{10}, H_6^3, H_6^2 H_{10}, H_6^4, H_6^3 H_{10}.$$

Every rule-minus-sphere residual is exactly nonzero. Thus the rule fails at that degree, proving that its algebraic degree is exactly t_f .

For $f = 9$, the twelve orbit representatives, in certificate order, are

$$(9, 0, 0), (8, 1, 0), (7, 2, 0), (7, 1, 1), (6, 3, 0), (6, 2, 1), \\ (5, 4, 0), (5, 3, 1), (5, 2, 2), (4, 4, 1), (4, 3, 2), (3, 3, 3).$$

Their orbit sizes are

$$12, 60, 60, 60, 60, 120, 60, 120, 60, 60, 120, 20,$$

which sum to 812. The eleven invariant rows are indexed by

$$(r, s) = (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (3, 0), (0, 2), (2, 1), (4, 0), (1, 2), (3, 1).$$

Exact row reduction gives

$$\text{rank } M_9 = \text{rank}[M_9 \mid b_9] = 11, \quad W(t) = W^{\text{part}} + tW^{\text{null}},$$

normalized so that $W_{333}^{\text{part}} = 0$ and $W_{333}^{\text{null}} = 1$. Substitution of the rational parameter $t = 1/27$ makes the twelve masses approximately

$$(0.028153, 0.006601, 0.053698, 0.115340, 0.099822, 0.101703, \\ 0.059136, 0.134511, 0.108995, 0.105497, 0.149506, 0.037037).$$

The displayed decimals are only orientation: the verifier substitutes $1/27$ into the exact affine vector over $\mathbb{Q}(\sqrt{5})$, obtains $M_9 W = b_9$ and $\sum W = 1$ identically, and proves all twelve signs by integer comparisons. In particular, the exact spherical targets include

$$\int_{\mathbb{S}^2} H_6^3 d\sigma = \frac{836352}{104975}, \quad \int_{\mathbb{S}^2} H_6^4 d\sigma = \frac{910787328}{15884375}.$$

At the first omitted invariant degree, 30, both residuals are exactly nonzero:

$$Q_9[H_{10}^3] - \int_{\mathbb{S}^2} H_{10}^3 d\sigma \approx 0.09380643058, \\ Q_9[H_6^5] - \int_{\mathbb{S}^2} H_6^5 d\sigma \approx -0.2197160741.$$

Their full exact $\mathbb{Q}(\sqrt{5})$ values, together with W^{part} and W^{null} , are regenerated in the certificate. Hence the selected frequency-nine rule has exact degree 29. $\square$

4.1 The complete frequency-nine positive family

Write the exact affine family from the proof as

$$W_\alpha(t) = A_\alpha + tN_\alpha, \quad A_\alpha = W_\alpha^{\text{part}}, \quad N_\alpha = W_\alpha^{\text{null}},$$

with $W_{333}(t) = t$. The exact, generally large, $\mathbb{Q}(\sqrt{5})$ coordinates A_α, N_α are in the embedded certificate and the machine-readable affine-family CSV.

Proposition 4.2 (Exact positivity interval). *Define*

$$t_L = -\frac{A_{900}}{N_{900}}, \quad t_U = -\frac{A_{810}}{N_{810}}.$$

Then every one of the twelve frequency-nine orbit masses is strictly positive if and only if

$$t_L < t < t_U.$$

Both endpoints lie in $\mathbb{Q}(\sqrt{5})$, and

$$t_L \approx 0.0261933635545922987, \\ t_U \approx 0.0380455174948779754, \\ t_U - t_L \approx 0.0118521539402856767.$$

At the lower wall $W_{900} = 0$; at the upper wall $W_{810} = 0$.

Proof. Each condition $A_\alpha + tN_\alpha > 0$ is an open half-line whose wall is $-A_\alpha/N_\alpha$. Exact sign comparisons in $\mathbb{Q}(\sqrt{5})$ show that $N_{900} > 0$, every lower-facing wall is at most t_L , $N_{810} < 0$, and every upper-facing wall is at least t_U . All twelve masses are positive at the exact midpoint $(t_L + t_U)/2$. Their intersection is therefore precisely (t_L, t_U) . This is a finite algebraic proof; the printed decimals locate the exact endpoints but do not decide any inequality. $\square$

The rational witness $t = 1/27 \approx 0.0370370370$ lies strictly inside this interval, near the upper wall. Consequently W_{810} is its smallest orbit mass. The witness is convenient, but it is neither unique nor the most balanced member of the family.

Proposition 4.3 (Maximum minimum total orbit mass). *The unique parameter maximizing the least total orbit mass is*

$$t_{\text{orb}} = \frac{A_{810} - A_{900}}{N_{900} - N_{810}} \in \mathbb{Q}(\sqrt{5}),$$

where the two active masses agree:

$$W_{900}(t_{\text{orb}}) = W_{810}(t_{\text{orb}}) = z_{\text{orb}}.$$

Numerically,

$$t_{\text{orb}} \approx 0.0346795431739537632, \quad z_{\text{orb}} \approx 0.0220321862949960231.$$

The other ten orbit masses are strictly larger.

Proof. The exact slopes satisfy $N_{900} > 0 > N_{810}$. For $t \leq t_{\text{orb}}$, the minimum of all masses is at most $W_{900}(t) \leq z_{\text{orb}}$. For $t \geq t_{\text{orb}}$, it is at most $W_{810}(t) \leq z_{\text{orb}}$. Exact substitution at t_{orb} proves that these two masses equal $z_{\text{orb}} > 0$, the other ten exceed it, the masses sum to one, and all moment residuals remain zero. Hence the bound is attained uniquely. $\square$

At this orbit-balanced point, the masses in certificate order are

$$(0.022032186, 0.022032186, 0.056642351, 0.098361995, \\ 0.090169215, 0.115493629, 0.063869857, 0.141775101, \\ 0.100125836, 0.097634012, 0.157184089, 0.034679543).$$

Remark 4.4 (Orbit balance is not node balance). The cubature formula applies the per-node weight $w_\alpha(t) = W_\alpha(t)/|\mathcal{O}_\alpha|$. Since the orbit sizes differ, Proposition 4.3 does not maximize the minimum per-node weight. That distinct optimum is determined by

$$\frac{W_{900}(t_{\text{node}})}{12} = \frac{W_{810}(t_{\text{node}})}{60}, \quad t_{\text{node}} = \frac{A_{810} - 5A_{900}}{5N_{900} - N_{810}} \in \mathbb{Q}(\sqrt{5}).$$

Exact comparison with all other per-node affine functions gives

$$t_{\text{node}} \approx 0.0301663175640353478, \quad \max_t \min_\alpha w_\alpha(t) \approx 0.000859562908985490228.$$

The same opposite-slope argument proves global optimality. Thus the paper exports three useful exact choices: the simple rational witness $1/27$, the orbit-mass-balanced point t_{orb} , and the node-weight-balanced point t_{node} . Their exact values and all twelve resulting weights are included in the affine-family CSV.

4.2 Compact exact formulas for $f \leq 4$

The first four rules fit legibly in the text. The orbit masses are

$$\begin{aligned} W_{100} &= 1, \\ W_{200} &= \frac{5}{21}, \quad W_{110} = \frac{16}{21}, \\ W_{300} &= \frac{26375 - 4500\sqrt{5}}{201432}, \\ W_{210} &= \frac{1570589 - 397404\sqrt{5}}{1007160}, \\ W_{111} &= \frac{-28971 + 17496\sqrt{5}}{41965}, \end{aligned}$$

and

$$\begin{aligned} W_{400} &= \frac{70025 - 12000\sqrt{5}}{801801}, \\ W_{310} &= \frac{3058238 - 84270\sqrt{5}}{8819811}, \\ W_{220} &= \frac{18302 - 270\sqrt{5}}{89089}, \\ W_{211} &= \frac{1059800 + 81000\sqrt{5}}{2939937}. \end{aligned}$$

The exact first-failure residuals are

$$\frac{132}{25}, \quad -\frac{4199}{5250}, \quad -\frac{14784}{35425}, \quad \frac{627(391241 - 123930\sqrt{5})}{374245000},$$

for $f = 1, 2, 3, 4$, respectively. The complete exact coefficients and failure residuals for $f = 5, 6, 7, 8, 9$ are preserved in the certificate and CSV identified in Section 10; decimals in the table are expository only.

4.3 Ready-to-use rule exports

Section 2 gives the node generator. Once duplicates are identified and the nodes are grouped into their H_3 -orbits, an implementation evaluates

$$Q_f[g] = \sum_{\alpha \in \Lambda_f} W_\alpha \left(\frac{1}{|\mathcal{O}_\alpha|} \sum_{x \in \mathcal{O}_\alpha} g(x) \right) = \sum_{x \in X_f} w_x g(x), \quad w_x = \frac{W_\alpha}{|\mathcal{O}_\alpha|}.$$

This distinction between total orbit mass and per-node weight is essential.

For example, the complete degree-eleven $f = 3$ rule is:

orbit	size	exact total mass W	W , decimal	per-node w_x
(3, 0, 0)	12	$(26375 - 4500\sqrt{5})/201432$	0.080983628	0.006748636
(2, 1, 0)	60	$(1570589 - 397404\sqrt{5})/1007160$	0.677118473	0.011285308
(1, 1, 1)	20	$(-28971 + 17496\sqrt{5})/41965$	0.241897899	0.012094895

Generating the 12 vertex nodes, 60 nonmidpoint edge nodes, and 20 face centers and applying these per-node weights integrates every spherical polynomial through degree 11; the exact degree-twelve residual is nonzero.

For orientation, the rational frequency-nine witness $t = 1/27$ exports as:

orbit	size	total mass W	per-node w_x
(9, 0, 0)	12	0.028152814	0.002346068
(8, 1, 0)	60	0.006601069	0.000110018
(7, 2, 0)	60	0.053698459	0.000894974
(7, 1, 1)	60	0.115340359	0.001922339
(6, 3, 0)	60	0.099822105	0.001663702
(6, 2, 1)	120	0.101703247	0.000847527
(5, 4, 0)	60	0.059136054	0.000985601
(5, 3, 1)	120	0.134511232	0.001120927
(5, 2, 2)	60	0.108995143	0.001816586
(4, 4, 1)	60	0.105496910	0.001758282
(4, 3, 2)	120	0.149505571	0.001245880
(3, 3, 3)	20	0.037037037	0.001851852

These decimals are deployment aids only. The embedded CSV contains exact $\mathbb{Q}(\sqrt{5})$ expressions for the witness, both conditioning optima, and every per-node division.

4.4 Exact sign decisions

For completeness, if $w = a + b\sqrt{5}$, the verifier never decides its sign numerically. When a and b have opposite signs, it compares a^2 and $5b^2$ in $\mathbb{Q}$, with the direction selected from the known rational signs of a and b . Thus all strict inequalities in Theorem 4.1 reduce to finite integer comparisons.

5 The exact OMEGA coefficient-field representation

The cubature calculation natively produces coefficients in $\mathbb{Q}(\sqrt{5})$. The following result provides an alternate exact representation of that field. It does not change the derivation in Sections 2–4.

Theorem 5.1 (Quadratic-field identification). *Let T be an indeterminate and let λ denote the residue class of the polynomial variable in*

$$K_\Omega = \mathbb{Q}(T)[\lambda]/(\lambda^2 - T\lambda - T^2).$$

Then K_Ω is a field and

$$r := \frac{2\lambda}{T} - 1$$

satisfies $r^2 = 5$. The map

$$\iota : \mathbb{Q}(T, \sqrt{5}) \longrightarrow K_\Omega, \quad \iota(\sqrt{5}) = r,$$

fixing $\mathbb{Q}(T)$, is an isomorphism. In particular,

$$\boxed{\iota(a + b\sqrt{5}) = (a - b) + \frac{2b}{T}\lambda}, \quad a, b \in \mathbb{Q}.$$

Proof. The discriminant of $X^2 - TX - T^2$ is $5T^2$, which is not a square in $\mathbb{Q}(T)$; hence the polynomial is irreducible and the quotient is a field. In the quotient,

$$\left(\frac{2\lambda}{T} - 1\right)^2 - 5 = \frac{4}{T^2}(\lambda^2 - T\lambda - T^2) = 0.$$

Thus $\sqrt{5} \mapsto r$ defines an embedding. Conversely,

$$\lambda = \frac{T(r+1)}{2},$$

so r generates the quotient over $\mathbb{Q}(T)$, proving surjectivity. Expanding $a + br$ gives the displayed formula. $\square$

The frozen OMEGA specialization uses

$$\Omega_0 = \varphi + \pi, \quad \tau_\Omega = \frac{1}{\Omega_0}, \quad \lambda_\Omega = \frac{\varphi}{\Omega_0} = \varphi\tau_\Omega.$$

At $(T, \lambda) = (\tau_\Omega, \lambda_\Omega)$,

$$\frac{2\lambda}{T} - 1 = 2\varphi - 1 = \sqrt{5},$$

so the representation recovers the ordinary positive real square root.

Corollary 5.2 (Exact translation of every certified weight). *Every orbit mass in Theorem 4.1 has an exact two-channel OMEGA normal form. For example,*

$$W_{300} = \frac{26375 - 4500\sqrt{5}}{201432}$$

becomes

$$W_{300} = \frac{30875T - 9000\lambda}{201432T}.$$

All 52 certified orbit masses are exported in both forms by the exact certificate.

Remark 5.3 (Direction of dependence). The logical arrows are

$$\begin{aligned} \text{Class-I geometry} &\longrightarrow \text{invariant moment system} \\ &\longrightarrow \text{positive coefficient in } \mathbb{Q}(\sqrt{5}) \\ &\longrightarrow \text{OMEGA normal form.} \end{aligned}$$

There is no arrow from Ω_0 , T , or λ_Ω back to the icosahedral moment equations.

6 Finite 15-bit replay and the first disagreement

The same frozen contract declares

$$\begin{aligned} S &= 10^{15}, & Q &= 339949771344778, & q &= \frac{Q}{S}, \\ D &= 10^{60}, \\ L_{60} &= 339949771344778929576702809321279193551009905014259416689444, \\ U_{60} &= L_{60} + 1, \end{aligned}$$

and the strict enclosure

$$L := \frac{L_{60}}{D} < \lambda_\Omega < \frac{U_{60}}{D} =: U.$$

Only this last strict enclosure is assumed in the two results below.

Proposition 6.1 (Unsigned 15-bit conformance). *Conditional on $L < \lambda_\Omega < U$,*

$$\lfloor \lambda_\Omega d \rfloor = \left\lfloor \frac{Qd}{S} \right\rfloor \quad (0 \leq d < 2^{15}).$$

Proof. Exact integer division checks, for all 32,768 values,

$$\left\lfloor \frac{Qd}{S} \right\rfloor = \left\lfloor \frac{L_{60}d}{D} \right\rfloor = \left\lfloor \frac{U_{60}d}{D} \right\rfloor.$$

Every number strictly between the two endpoints has the same floor. The computation is finite, exhaustive, and uses arbitrary-precision integers. $\square$

The package also checks the implementation boundary

$$Q \cdot 32767 < 2^{64} - 1, \quad Q \cdot 65535 > 2^{64} - 1,$$

which explains why the declared unsigned state width is 15 bits when the product is required to fit in an unsigned 64-bit word.

Theorem 6.2 (First disagreement, with minimality). *Conditional on $L < \lambda_\Omega < U$, the least nonnegative integer d for which*

$$\lfloor \lambda_\Omega d \rfloor \neq \left\lfloor \frac{Qd}{S} \right\rfloor$$

is

$$\boxed{d_* = 42,189,065}.$$

At this value,

$$\left\lfloor \frac{Qd_*}{S} \right\rfloor = 14,342,162, \quad \lfloor \lambda_\Omega d_* \rfloor = 14,342,163.$$

Proof. Exact cross multiplication gives the strict chain

$$\frac{846547}{2490212} < \frac{Q}{S} < \frac{14342163}{42189065} < L < \lambda_\Omega < U < \frac{13495616}{39698853}.$$

Moreover,

$$14342163 \cdot 2490212 - 846547 \cdot 42189065 = 1$$

and

$$13495616 \cdot 42189065 - 14342163 \cdot 39698853 = 1.$$

Thus the middle fraction is a Farey neighbor of the fraction on each side. Any reduced rational strictly between two determinant-one neighbors has denominator at least the sum of their denominators. Therefore

$$\frac{14342163}{42189065}$$

has the smallest denominator among all rationals in $(Q/S, U)$: any other rational in that interval lies in one of the two adjacent Farey intervals and has denominator strictly larger than 42,189,065.

If a floor disagreement occurred at a smaller positive d , the integer $m = \lfloor Qd/S \rfloor + 1$ would give, after reduction if necessary, a rational $m/d \in (Q/S, \lambda_\Omega] \subset (Q/S, U)$ with denominator below d_* , a contradiction.

At d_* , the middle numerator 14,342,163 lies strictly between $(Q/S)d_*$ and Ld_* , while the upper endpoint still has floor 14,342,163. Hence the two displayed floors follow. $\square$

6.1 Exact Julia replay kernel

For proof-level replay, a direct floating-point evaluation of $\varphi/(\varphi + \pi)$ is unnecessary and would make correctness depend on a precision analysis. Listing 1 instead compares the rational implementation with both published decimal endpoints using only arbitrary-precision integers. The strict enclosure then supplies the real number between them. The same kernel checks the determinant-one Farey data and the W_{300} channel conversion. It remains logically independent of the cubature moment systems.

The Version 1.1 listing used the constructor form `BigInt(...)`, which raises a `MethodError` under Julia 1.11.6. Version 1.2 uses `parse(BigInt,...)` for decimal string literals. Version 1.3 leaves the kernel and mathematical statements unchanged; it corrects only the embedded reproducibility layout. An independently supplied Julia 1.11.6 execution of the corrected kernel reported **EXACT OMEGA REPLAY: PASS**, covering the 32,768 endpoint floor identities, unsigned-width bounds, both Farey determinants, and the W_{300} channel identity. The execution record is embedded for provenance; the mathematical assertions are also regenerated by the exact Python checks described in Section 10.

Listing 1: Exact Julia reference kernel for OMEGA interoperability and replay.

```

1 module ExactOmegaReplay
2
3 # Exact reference kernel for the coefficient map and frozen floor contract.
4 # No floating-point evaluation of pi or lambda_Omega is used.
5
6 struct QSqrt5
7     a::Rational{BigInt}
8     b::Rational{BigInt}
9 end
10
11 "Return A,B for a+b*sqrt(5) -> A+B*(lambda/T)."
12 function omega_channels(x::QSqrt5)
13     return (x.a - x.b, 2 * x.b)
14 end
15
16 const S = BigInt(10)^15
17 const Q = parse(BigInt, "339949771344778")
18 const D = BigInt(10)^60
19 const L60 = parse(BigInt,
20     "339949771344778929576702809321279193551009905014259416689444"
21 )
22 const U60 = L60 + 1
23
24 function verify_15bit_contract()
25     for d in BigInt(0):BigInt(32767)
26         qfloor = fld(Q * d, S)
27         @assert qfloor == fld(L60 * d, D)
28         @assert qfloor == fld(U60 * d, D)
29     end
30     @assert Q * BigInt(32767) < BigInt(typemax(UInt64))
31     @assert Q * BigInt(65535) > BigInt(typemax(UInt64))
32     return true
33 end
34
35 function verify_first_disagreement()
36     left_n, left_d = BigInt(846547), BigInt(2490212)
37     nstar, dstar = BigInt(14342163), BigInt(42189065)
38     right_n, right_d = BigInt(13495616), BigInt(39698853)
39
40     # left < Q/S < nstar/dstar < L < lambda < U < right
41     @assert left_n * S < Q * left_d
42     @assert Q * dstar < nstar * S
43     @assert nstar * D < L60 * dstar
44     @assert U60 * right_d < right_n * D
45
46     # Determinant-one neighboring fractions certify denominator minimality.
47     @assert nstar * left_d - left_n * dstar == 1
48     @assert right_n * dstar - nstar * right_d == 1
49

```

```

50     @assert fld(Q * dstar, S) == nstar - 1
51     @assert fld(L60 * dstar, D) == nstar
52     @assert fld(U60 * dstar, D) == nstar
53     return true
54 end
55
56 function verify_weight_example()
57     w300 = Qsqrt5(big"26375" // big"201432",
58                 -big"4500" // big"201432")
59     A, B = omega_channels(w300)
60     @assert A == big"30875" // big"201432"
61     @assert B == -big"9000" // big"201432"
62     return true
63 end
64
65 function main()
66     @assert verify_15bit_contract()
67     @assert verify_first_disagreement()
68     @assert verify_weight_example()
69     println("EXACT OMEGA REPLAY: PASS")
70 end
71
72 end # module
73
74 ExactOmegaReplay.main()

```

7 The all-frequency convex formulation

The verified frequencies suggest the following deliberately limited statement.

Conjecture 7.1 (Positive fixed-Class-I cubature). For every $f \geq 1$, there are strictly positive H_3 -orbit masses on X_f whose rule is exact for every spherical polynomial of degree at most

$$t_f = \begin{cases} 3f + 2, & f \text{ odd}, \\ 3f + 3, & f \text{ even}. \end{cases}$$

The conjecture makes no maximality claim. It is proved here only for $1 \leq f \leq 9$; it remains open here for $f \geq 10$.

Frequency nine is the first nonsquare instance. Section 4.1 now resolves its complete positive affine family and both max–min choices exactly; the decimals there are consequences of exact $\mathbb{Q}(\sqrt{5})$ identities, not feasibility evidence.

The next case remains research rather than a claim. Direct combinatorial counting gives

$$|X_{10}| = 1002, \quad t_{10} = 33, \quad |\Lambda_{10}| = 14, \quad \#\{(r, s) : 6r + 10s \leq 33\} = 14.$$

Thus frequency ten produces a square 14-by-14 invariant system. If it has full rank, its candidate rule is unique; its exact signs would then decide whether the proposed positive sequence continues at $f = 10$. No frequency-ten matrix inversion or positivity result is asserted here.

Proposition 7.2 (Relative-interior formulation). *Fix f and t . Let v_α be the invariant moment vectors from the proof of Theorem 4.1, including the constant coordinate, and let b be the spherical target vector. Strictly positive orbit masses exact through degree t exist if and only if*

$$b \in \text{relint conv}\{v_\alpha : \alpha \in \Lambda_f\}.$$

Proof. A positive orbit-mass vector expresses b as a convex combination using every v_α with positive coefficient. Any supporting affine functional that is constant at b must then be constant at every v_α , so b is in the relative interior.

Conversely, let c be the average of all v_α . If b is in the relative interior, then for sufficiently small $\varepsilon > 0$,

$$y = \frac{b - \varepsilon c}{1 - \varepsilon}$$

still lies in the convex hull. Combining any convex representation of y with the uniform positive representation of c gives a representation of b in which every coefficient is positive. $\square$

This converts the all-frequency problem into an exact finite-dimensional convex-feasibility problem and its separating-hyperplane dual. Numerical linear programming may guide the search, but it is not a proof unless its output is promoted to exact moment and sign certificates.

8 Compatibility with the Papers I–V global object

The multi-sorted construction of [7] is written here as

$$\mathfrak{G} = (\mathcal{S}_{\text{SRSD}}, \mathcal{A}_\Omega, \mathcal{D}_{\text{quant}}, \mathcal{D}_{\text{rat}}, \mathcal{H}_{\text{ico}}, \mathcal{C}_{\text{fin}}; \mathcal{J}).$$

Its icosahedral sort already contains the node sets X_f , their equal-weight measures, and the radial limiting measure. Its arithmetic sort contains the distinguished OMEGA field. The present cubature data attach to those two sorts, and to the exact SRSD coordinate layer, without identifying the distinct measures or dynamics.

8.1 Same nodes, distinct measures

The radial branch of $\mathfrak{G}$ uses

$$\mu_f = \frac{1}{|X_f|} \sum_{x \in X_f} \delta_x.$$

The certified orbit masses instead define

$$\nu_f = \sum_{\alpha \in \Lambda_f} \frac{W_{f,\alpha}}{|\mathcal{O}_{f,\alpha}|} \sum_{x \in \mathcal{O}_{f,\alpha}} \delta_x, \quad 1 \leq f \leq 9.$$

Every $W_{f,\alpha}$ is strictly positive and their sum is one. Thus ν_f and μ_f have the same full support X_f , but generally assign different masses to its orbits.

Proposition 8.1 (Exact cancellation of the first radial invariants). *For the invariant harmonics of Section 3,*

$$\nu_f(H_6) = 0 \quad (2 \leq f \leq 9), \quad \nu_f(H_{10}) = 0 \quad (3 \leq f \leq 9).$$

The remaining low-frequency values are

f	$\nu_f(H_6)$	$\nu_f(H_{10})$
1	132/25	988/625
2	0	−4199/5250.

Proof. The normalized spherical means of H_6 and H_{10} are zero. Theorem 4.1 is exact through degree at least six for $f \geq 2$ and through degree at least ten for $f \geq 3$, which gives the asserted zeros. The two exceptional rows are exact evaluations reproduced by the interface verifier. $\square$

The distinction is visible already at $f = 2$:

observable	spherical mean	equal-weight μ_2	cubature ν_2
1	1	1	1
z^2	1/3	1/3	1/3
H_6	0	33/100	0

The first two rows agree because both probability measures are fully icosahedrally invariant; the degree-six row is where orbit reweighting has new effect.

This is the sharp measure-theoretic distinction between the two papers. The radial asymptotic formula and the positive limiting value $\mu_{\text{rad}}(H_6) = C_0 > 0$ in [7] remain statements about μ_f and μ_{rad} ; they are not inherited by the reweighted measures ν_f . Orbit reweighting cancels the obstruction on the fixed nodes rather than altering the radial geometry.

8.2 Arithmetic specialization and canonical records

The global object writes

$$\tau_0 = \frac{1}{\Omega_0}, \quad \lambda_\Omega = \frac{\varphi}{\Omega_0}, \quad \lambda_\Omega^2 - \tau_0 \lambda_\Omega - \tau_0^2 = 0.$$

The generic quotient of Theorem 5.1 specializes by

$$T \mapsto \tau_0, \quad \lambda \mapsto \lambda_\Omega.$$

This specialization does not collapse the coefficient field: τ_0 is transcendental, since algebraicity of τ_0 would make $1/\tau_0 - \varphi = \pi$ algebraic. Consequently the generic identity becomes the exact global-object identity

$$\sqrt{5} \mapsto \frac{2\lambda_\Omega}{\tau_0} - 1, \quad a + b\sqrt{5} \mapsto (a - b) + \frac{2b}{\tau_0} \lambda_\Omega.$$

Every certified orbit mass therefore lies in the arithmetic sort $\mathcal{A}_\Omega$, and the canonical symbolic record Can_Ω specified in [7] applies to it. The verifier in this package certifies the algebraic normal forms; it does not silently substitute a new implementation for that source's frozen byte serialization.

8.3 Exact transport through SRSD coordinates

For a positive scale vector $s \in (0, \infty)^3$, let E_s and D_s be the exact encoder and decoder of [7]. Explicitly,

$$E_s(y) = (q_s(y), r_s(y)), \quad D_s(d, \rho) = \text{diag}(s)d + \rho, \quad D_s E_s = \text{id}_{\mathbb{R}^3}.$$

Define

$$\widehat{\nu}_{f,s} = (E_s)_\# \nu_f.$$

Then every integrable observable Y satisfies

$$\int Y(D_s e) d\widehat{\nu}_{f,s}(e) = \int_{\mathbb{S}^2} Y(x) d\nu_f(x).$$

In particular, for every spherical polynomial p of degree at most t_f ,

$$\int p(D_s e) d\widehat{\nu}_{f,s}(e) = \int_{\mathbb{S}^2} p d\sigma.$$

This is lossless coordinate transport of an already proved rule. Dropping a residual channel, rounding, or retaining only an SRSD quotient is a different record map and is not covered by the identity.

Theorem 8.2 (Conservative cubature extension). *Let $\mathcal{C}_{\text{cub}}$ contain the fifty-two orbit masses, the measures ν_f , their exact degrees, and their first-failure residuals for $1 \leq f \leq 9$. Adjoining this sort and the three displayed interfaces gives*

$$\mathfrak{G}^{\text{cub}} = (\mathfrak{G}, \mathcal{C}_{\text{cub}}; \mathcal{J} \cup \mathcal{J}_{\text{cub}}).$$

This extension exists and is conservative in the following concrete sense: projection onto $\mathfrak{G}$ recovers every original component and gluing law unchanged, while projection onto the Class-I geometry and $\mathcal{C}_{\text{cub}}$ recovers Theorem 4.1. No floor dynamics, finite-limb record, or SRSD transport statement is used to solve the cubature moment systems.

Proof. The exact certificate constructs $\mathcal{C}_{\text{cub}}$ over $\mathbb{Q}(\sqrt{5})$, proves positivity and the moment identities, and evaluates every first failure. The arithmetic interface is Theorem 5.1 followed by the specialization above. The SRSD interface follows from $D_s E_s = \text{id}$ and the definition of pushforward measure. These add a new sort and new maps whose codomains are existing sorts; they alter none of the data or equations already present in $\mathfrak{G}$. The two stated projections and the dependency claim follow directly. $\square$

The complete dependency boundary for this single integrated construction is summarized below.

Layer	Exact interface used	Not claimed
Radial	same X_f ; μ_f and ν_f distinguished	radial asymptotics for ν_f
OMEGA	exact normal form for every $W_{f,\alpha}$	OMEGA origin of geometry or weights
SRSD	lossless decoded moment transport	compression or quotient-only exactness
Floor systems	15-bit replay remains intact	role in moment equations or positivity
Finite records	the frozen canonical record accepts each field element	a replacement byte serialization
Cubature	exact rules and first failures for $f \leq 9$	proof for $f \geq 10$, maximality, novelty, or priority

9 Provenance and literature boundary

Invariant cubature and icosahedral orbit quadrature are classical subjects. McLaren [4], Goethals and Seidel [2], Konyaev [3], Ahrens and Beylkin [1], Nozaki and Sawa [8], and Popov [9, 10] are among the relevant starting points. Ahrens–Beylkin and Popov construct invariant rules by solving for or optimizing node generators; their supports are not fixed in advance to the complete radial Class-I barycentric grids used here. The present question is narrower: positive orbit-constant weights on those already fixed node sets. Wang and Lee’s construction of projected icosahedral–hexagonal grids supplies additional context for the geometry, but not these orbit-mass formulas [11].

McLaren’s classical 72-point degree-14 formula should not be identified with any X_f : the equation $10f^2 + 2 = 72$ has no integral solution. At $f = 1$, the twelve icosahedral vertices and their degree-five exactness are classical. At $f = 2$, X_2 consists of the twelve original vertices together with thirty edge midpoints; only the midpoint suborbit is the vertex set of the icosidodecahedron.

No priority conclusion is drawn here. The $f = 1$ icosahedral design and the $f = 2$ 42-node rule should be treated as classical or implicit unless a complete historical audit establishes otherwise. A targeted search completed on 23 August 2026 did not locate an explicit positive orbit-constant rule on the precise fixed grids X_f for $3 \leq f \leq 9$, nor the displayed rank-eleven frequency-nine affine family. Failure to locate a source is not evidence of priority: a specialist cubature-history audit remains necessary before any novelty language is considered.

The radial construction and degree-six normalization are imported from [5]. The OMEGA contract, algebra, and enclosure are frozen from [6]; the live-site snapshot is provenance only and is not an

executable dependency. The ambient multi-sorted notation, exact SRSD transport law, and the distinction between the equal-weight radial measure and other attached sorts are imported from the Papers I–V synthesis [7]. Its local PDF snapshot is frozen by hash; none of its unrelated dynamical or foundational claims is needed for the cubature proof.

10 Reproducibility

The reproducibility record contains the manuscript source, the frozen contract, the native baseline source, exact scripts, machine-readable coefficients, and human-readable certificates. The principal frozen artifacts are:

Artifact	SHA-256
certificates/cubature_f1_f9_certificate.txt	27eeea586a289f196ee5a44aef09ac1e b416eaa18fa44cb8c05eba0a325ba6bb
data/orbit_masses_f1_f9.csv	eed325c55fe183b36b46b591f758935b cf748a91c2427a09a965337895354b98
data/f9_affine_family.csv	851a5ff03452e21f7839285cd6342338 5c936907568e852a174056197c20a95f
certificates/omega_embedding_certificate.txt	d545a86d15fde3cf4c29ca4010e1f63 e46096b5a77522d9a94d3d6e72e27bcf1
certificates/omega_15bit_bridge_certificate.txt	4ef4248b51b69f9776fbeb375c3d99d 793db153a7e3bea3d25175672d8e2f0ac
omega/OMEGA_NAUGHT_1_0.txt	ad71b4d5b9395ab47fae8e78decbbab 56aee760c9b20833358b8940eab68f8b8
certificates/global_object_interface_certificate.txt	da57a2e5c931d0b12781806a02d9e41e 12c2be2929daa893456c00e049ebaeb9
src/exact_omega_replay.jl	e8983068f4243589c610ac74ca689394 dffc8a1b7127fb2cb9e85688b280fff4
certificates/julia_1_11_6_execution_record.txt	c27e3460b1ff9b47b19c5d853242e30 9ac71cf70ea1c67817fdec0ebd2707623
global_object/Global_Mathematical_Object_Papers_I-V_v1.0.pdf	cbe30136476c14602545eaab3b0984d0 05cce6626378ddf8c402e28363c90ed2

To preserve the requested single-document form, the exact CSV, certificates, verification sources, frozen contract, Julia kernel, requirements file, run script, manuscript source, and the hash-pinned global-object source PDF are embedded as attachments in this PDF. A PDF viewer with an attachments pane can extract them all into one directory. Version 1.3 makes the runner and verifiers detect that flat attachment layout as well as the original project directory layout. They are reproducibility payloads of this paper, not separate papers or companion claims.

With Python 3 and SymPy 1.14.0, extract all attachments into one directory and run

```
bash run_checks.sh.
```

The first script reconstructs all moment systems and weights rather than trusting the stored CSV. It verifies harmonicity, exact spherical moments, orbit counts, the eight square inversions, the rank-eleven frequency-nine affine system, its complete positive interval, its $1/27$ witness, and both exact max–min rules; it also checks zero residuals, strict signs, first failures, per-node exports, the frequency-ten combinatorial counts, and the OMEGA channel export. The second verifies the quotient-algebra identity and all 52 specializations. The third exhausts the 15-bit domain and checks the Farey minimality certificate. The fourth verifies the global-object interface: common support, probability normalization, exact obstruction cancellation, arithmetic specialization, and preservation of the source-PDF hash.

The Julia file is a compact independent replay kernel rather than an input to `run_checks.sh`. Its Version 1.11.6 execution record and the exact Python regeneration of the same integer identities are both embedded, so the external runtime report is clearly separated from the proof decisions made by the deterministic suite.

The paper compiles with three runs of `pdflatex`; three runs are used so the table of contents and all cross-references are settled.

11 Conclusion

The fixed radial Class-I grids support a concrete sequence of exact positive cubature rules through frequency nine. Their proof is native to icosahedral geometry and invariant moment theory. At frequency nine the result is a complete open interval of exact positive rules, with separate closed-form choices for balancing orbit masses and per-node weights. The OMEGA quotient algebra then gives every resulting $\mathbb{Q}(\sqrt{5})$ coefficient an exact alternate two-channel representation, while the frozen 15-bit contract supplies an independent deterministic replay test with a proved first disagreement.

Relative to the Papers I–V global object, the result is a conservative new cubature sort. It reweights the existing Class-I node sets, embeds every weight in the existing OMEGA arithmetic sort, and survives exact SRSD coordinate transport. The equal-weight radial measure, its asymptotics, and the rational and real floor dynamics retain their original meanings.

This is the intended integration: exact geometry first, exact representation second, and explicit dependency boundaries throughout.

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