Exact Arithmetic Certification of Geometric Reconstruction and Transport

Paper VIII of the OMEGA series · Final manuscript

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19 September 2026

DOI: 10.5281/zenodo.22848213

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Abstract

An exact arithmetic certificate acquires geometric meaning through a proved representation map and a justified interpretation. We develop these interfaces for a fixed marked scale family, rational floor dynamics, spherical integration, and incompressible transport. Ordered Euclidean remainders recover every registered state, and retained scale and placement data recover the complete marked construction. Rational separators establish the first disagreement between the real and rational floor maps at seed 42,189,065; exact predecessor intervals describe all fibers and first-hitting layers. A homothety theorem converts the scalar rounding discrepancy into an exact Hausdorff distance. Finite moment identities prove integration for every polynomial in the stated degree class, with the twelve normalized icosahedral vertices providing a complete degree-five example and an exact degree-six obstruction. A volume-compensating deformation and a smooth localization realize any prescribed finite motion in the stated deformation family on a compact set in a smooth, finite-energy forced incompressible flow. The construction extends to every finite ambient dimension at least three. For radial localization, angular kinetic energy remains within degree-four cubature, whereas individual velocity second moments can require degree six. We also specify the data and local correctness obligations that permit these maps to compose, distinguish intrinsic predecessor recursion from optional geometric constructions, and release the arithmetic, interface, symbolic, and numerical code. The proofs establish the algorithms' mathematical meaning; the code reproduces the finite calculations and the declared models.

Keywords: exact arithmetic; geometric reconstruction; floor dynamics; Euclidean remainders; finite certificates; spherical cubature; incompressible transport.

1 Scope and mathematical contribution

The guiding problem is to explain how a geometric statement acquires an arithmetic representation, how finite verification establishes the relevant arithmetic facts, and why their interpretation in geometry is valid. These are separate proof obligations. An integer representation does not automatically retain a shape. A finite calculation does not automatically establish a statement outside its checked domain. A scalar invariant does not automatically determine the object carrying it.

The preceding papers already contain substantial parts of this account. Paper I proves the exact

remainder expansion and its quantization error bound [1, Theorems 3.1 and 5.1]. Paper II proves ordered reconstruction, maximal parameter intervals, the first floor disagreement, and finite tree data [2]. Paper III specifies exact cell and residual coordinates and their reconstruction conditions [3]. Paper V develops the scalar field and its exact representations [5]. Paper VII fixes the marked construction from which the selected lengths are measured [7]. Papers IV and VI use finite exact calculations to establish integration statements over continuous geometric domains [4,6]. The current geometric fluid extension supplies an explicit transport application [8].

The contribution is an explicit system of representation and interpretation theorems for these objects. Theorem 6.1 proves reconstruction of the marked scale family with its required retained data. Theorem 6.2 identifies its quantization discrepancy as an exact Hausdorff distance. Theorems 7.4 and 7.6 give smooth forced realizations in three and higher dimensions. Theorems 8.1–8.6 turn finite moments into universal polynomial identities and establish the precise reach of those identities after localization. Theorem 10.1 proves composition under local correctness hypotheses, and Theorem 10.2 assembles the resulting conclusions. Each branch retains its own assumptions; the common account is the compatibility of specified maps and theorems.

The general use of finite computation inside mathematical proof has established precedents. Decision procedures with proved correctness are central to proof by reflection [11]; formal geometric proofs and certified nonlinear inequalities provide further examples [12,13]. The exhaustive computation of Rokicki, Kociemba, Davidson, and Dethridge [10] is a methodological precedent for complete domain coverage, boundary analysis, and released verification code. Here the computational objects are integer intervals and ordered records. Standard terms are used throughout, and the classical spherical-design and iterated-function-system results are attributed explicitly [15–17]. No priority claim is made for the general principle of connecting finite certificates to universal theorems.

The paper is self-contained for the claims it proves. Related results from the supplied OMEGA corpus are cited for provenance and context. The electronic companion separates an exact certificate, a checked data interface, symbolic formulas, and numerical models. Its acceptance rules are justified by the mathematics below; execution provides reproducibility evidence at the scope stated in Section 11.

1.1 Notation and kinds of assertion

Write

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \varphi' = \frac{1-\sqrt{5}}{2}, \qquad \Omega_0 = \varphi + \pi, \qquad \tau = \Omega_0^{-1}, \qquad \lambda = \tau\varphi.$$

For the rational register, fix

$$Q = 339949771344778, \quad S = 10^{15}, \quad q = Q/S, \quad F(d) = \lfloor qd \rfloor, \quad D = \{0, \dots, 32767\}.$$

The real affine map is $T(v) = 1 + \lambda(v-1)$. We reserve F for the integer map and use $0 < q < \lambda < 1$. Exact integer arithmetic means unbounded integers and rational arithmetic, with no floating-point comparisons controlling acceptance. A finite certificate is data satisfying a specified finite predicate. A proof of that predicate's soundness explains what acceptance implies. An execution of a Python checker is evidence about those data; the present checker is not itself verified in a proof assistant.

All geometric statements below concern the explicit constructions and hypotheses given here. Smooth fluid realization is an existence result for a force constructed with the motion. It is not a conclusion about an arbitrary prescribed force or arbitrary initial data.

2 The marked geometry and its observations

Put

$$\varepsilon = \pi/648000, \qquad \delta = \varepsilon/2, \qquad s_\delta = \sin\delta, \qquad c_\delta = \cos\delta.$$

Choose a center $\alpha \in \mathbb{R}^3$ and an ordered orthonormal pair u, v. The placement data are $\kappa = (\alpha, u, v)$; the construction rules and the labels below are also fixed. At scale $a > 0$, set

$$p_a(\theta) = \alpha + a(\cos\theta\, u + \sin\theta\, v), \qquad m_a(t) = \alpha + a(c_\delta u + tv).$$

Define the closed parameter intervals

$$\Theta = [\delta, 2\pi - \delta], \qquad J_+ = [s_\delta, \varphi^2 + s_\delta], \qquad J_- = [-\varphi^2 - s_\delta, -s_\delta].$$

The marked outline is

$$\Gamma_\kappa(a) = p_a(\Theta) \cup m_a(J_+) \cup m_a(J_-).$$

Its six labeled points are $m_a(\pm x_j)$, where

$$x_0 = s_\delta, \qquad x_1 = 1 + s_\delta, \qquad x_2 = \varphi^2 + s_\delta.$$

The center and the opposite semicircle $p_a([\pi/2, 3\pi/2])$ are retained as marked data. Let G be the mean of the two outer straight lengths, between heights x_1 and x_2 and their negatives. Let C be the length of the retained opposite semicircle. These selection rules are part of the definition [7].

Proposition 2.1 (Measurements and geometric recovery). The marked construction satisfies

$$G = a\varphi, \qquad C = a\pi, \qquad (G + C)/a = \Omega_0, \qquad G/(G + C) = \lambda.$$

The scalar ratio $G/(G + C)$ is constant throughout this family and does not determine a or κ. With κ fixed, either a or G determines the entire marked construction. If α and the two labeled seed points $p_+ = m_a(s_\delta)$ and $p_- = m_a(-s_\delta)$ are retained, they recover a, u, v by

$$a = |p_+ - \alpha|, \quad u = \frac{p_+ + p_- - 2\alpha}{2ac_\delta}, \quad v = \frac{p_+ - p_-}{2as_\delta}.$$

Proof. The outer height difference is $\varphi^2 - 1 = \varphi$, and $|m_a(t) - m_a(t')| = a|t - t'|$. Also $|p_a'(\theta)| = a$, so the selected semicircle has length $a\pi$. The displayed ratios follow by cancellation. All scales and placements have the same ratio, proving its noninjectivity. Conversely, $a = G/\varphi$ and the defining parametrizations determine the construction when κ is fixed. The seed points have center-relative norm a because $c_\delta^2 + s_\delta^2 = 1$. Their sum and difference give the two frame formulas; their denominators are positive. □

The complete outline has component length $a(2\pi - \varepsilon) + 2a\varphi^2$. Consequently $G + C$ is a selected measurement, not the outline's full length. The prescribed golden extensions and choice of retained arc are essential construction data; the angle ε alone does not determine them.

Proposition 2.2 (Exact observation and lost distinctions). Let $H : X \to Y$ be an observation and let $E : Y \to Z$ be injective. Then

$$E(H(x)) = E(H(x')) \quad \Longleftrightarrow \quad H(x) = H(x').$$

In particular, an exact encoding of $H(x)$ permits recovery of x precisely when H is injective on the admitted source family, or when retained additional data make the combined observation injective.

Proof. One implication follows by applying E to an equality, and the other follows from injectivity of E. An inverse on the image exists exactly when every fiber contains at most one source element. Additional data replace H by the combined map, to which the same argument applies. □

This is the reconstruction criterion of Paper III [3] applied to a concrete geometry. It distinguishes recovery of the scale family in Proposition 2.1 from recovery of an arbitrary geometric object from one scalar.

3 Exact real coordinates and integer records

3.1 Cell and residual coordinates

Proposition 3.1 (Exact resolution coordinates). For $h > 0$, the maps

$$E_h(y) = (\lfloor y/h \rfloor, y - h \lfloor y/h \rfloor), \qquad D_h(d, r) = hd + r$$

are inverse bijections between $\mathbb{R}$ and $\mathbb{Z} \times [0, h)$. Coordinatewise, the same statement holds in $\mathbb{R}^m$ with prescribed positive scales. If $U : X \to X$, where $X \subseteq \mathbb{R}^m$, then

$$\widehat{U} = E_h U D_h \quad \text{on } E_h(X), \qquad \widehat{U}^n = E_h U^n D_h.$$

Proof. The floor inequality places the remainder in $[0, h)$ and gives $D_h E_h(y) = y$. Conversely $\lfloor (hd + r)/h \rfloor = d$ for $0 \le r < h$, so $E_h D_h(d, r) = (d, r)$. Apply the identity in each coordinate. In iterates, adjacent $D_h E_h$ factors cancel. □

This exact coordinate change retains a real residual. It does not encode every real state in a finite bit string. If scales vary, they must be retained as data: for example, $y = 1$ at scale 1 and $y = 2$ at scale 2 both produce the pair $(1, 0)$.

Discarding r is a different operation. Even the next cell of a continuously scaled state depends on it:

$$\left\lfloor \frac{\lambda(hd + r)}{h} \right\rfloor = \lfloor \lambda(d + r/h) \rfloor.$$

For $d = 2$, the values $r = 0$ and $r = 0.99h$ give different next cells. The autonomous integer rule $d \mapsto \lfloor \lambda d \rfloor$ therefore describes resetting the residual to zero at each step, not merely observing a continuous trajectory without intervention.

3.2 Admissible ordered remainders

Theorem 3.2 (Exact reconstruction and parameter interval). Let $0 < Q < S$ be integers and $q = Q/S$. For integers e, ρ, a transition record (e, ρ) is the quotient and remainder record of an integer

$d \geq 0$ exactly when

$$e \geq 0, \qquad 0 \leq \rho < S, \qquad Q \mid (Se + \rho), \qquad d = (Se + \rho)/Q.$$

For a finite orbit $d_{j+1} = \lfloor q d_j \rfloor$, $0 \leq j < k$, put $\rho_j = Q d_j - S d_{j+1}$. The terminal state and ordered remainders reconstruct every state, and

$$Q^k d_0 = S^k d_k + \sum_{j=0}^{k-1} S^j Q^{k-1-j} \rho_j.$$

The parameters $\beta \in (0, 1)$ preserving all these transitions form exactly

$$I = \bigcap_{j : d_j > 0} \left[\frac{d_{j+1}}{d_j}, \frac{d_{j+1} + 1}{d_j} \right) \cap (0, 1).$$

The empty intersection imposes no restriction beyond $(0, 1)$. This interval is maximal.

Proof. Euclidean division gives $Qd = Se + \rho$ and $0 \leq \rho < S$. Conversely the displayed divisibility defines a nonnegative integer d and gives $qd = e + \rho/S$, whose floor is e. Apply this inverse successively from $j = k - 1$ down to 0. Multiplying the one-step identity by powers of Q, S and substituting gives the finite sum, equivalently by induction on k. For a positive d_j, the equation $\lfloor \beta d_j \rfloor = d_{j+1}$ is equivalent to the corresponding half-open interval. A zero state imposes no restriction. Intersecting these necessary and sufficient conditions proves the claim and maximality. □

This is the ordered-remainder and parameter theorem of Paper II [2, Sections 4–8]. Bounds on remainders alone are insufficient: reverse divisibility, order, the terminal state, and the reconstructed seed must also be checked. For a complete orbit stopped at its first zero, each reconstructed predecessor is positive and larger than its successor. This excludes padding with extra zero transitions.

On D, exact intersection over all positive seeds gives the maximal interval

$$\left[\frac{4196}{12343}, \frac{7039}{20706} \right), \qquad \text{width } \frac{1}{255574158}.$$

The lower endpoint is attained at seed 12343 (also 24686); the upper endpoint is attained at 20706. The companion checks every defining inequality and both witnesses. There is also a short analytic proof of completeness. Put $\ell = 4196/12343$ and $u = 7039/20706$. Direct multiplication gives

$$7039 \cdot 12343 - 4196 \cdot 20706 = 1, \qquad \ell < q < u,$$

and the sum of the endpoint denominators is $33049 > 32767$. By Lemma 4.1, no fraction strictly between ℓ and u has a representation with positive denominator at most 32767. If some $\beta \in [\ell, u)$ changed a floor on D, an integer separator between β and q would lie strictly inside (ℓ, u), which is impossible. The lower endpoint itself does not cause a change: a separating fraction must be strictly greater than the smaller parameter. Conversely, at seed 12343 a parameter below ℓ gives a floor at most 4195 instead of 4196, and at seed 20706 a parameter at least u gives a floor at least 7039 instead of 7038. This proves maximality without enumerating the interval conditions. Invariance of D then gives identical complete forward orbits for every parameter in the interval. Thus even the complete transition table does not identify a unique real parameter.

4 Certifying the real and rational interface

The floor operation makes a small parameter error potentially decisive. The relevant bound is distance to an integer boundary, not the number of matching decimal places.

Lemma 4.1 (Rational separators). If $x < y$, then $\lfloor nx \rfloor < \lfloor ny \rfloor$ for a positive integer n exactly when an integer p satisfies $x < p/n \leq y$. If reduced fractions $a/b < c/d$ satisfy $bc - ad = 1$, every reduced fraction strictly between them has denominator at least $b + d$.

Proof. A separator gives $nx < p \leq ny$; conversely take $p = \lfloor nx \rfloor + 1$ when the floors differ. For the denominator assertion, suppose $a/b < p/n < c/d$. The positive integers $bp - an$ and $cn - dp$ satisfy

$$n = d(bp - an) + b(cn - dp) \geq d + b,$$

using $bc - ad = 1$. The same inequality applies to the reduced denominator of any separator. □

Theorem 4.2 (Exact floor agreement and its first failure). With the frozen constants,

$$\lfloor \lambda d \rfloor = F(d) \quad (0 \leq d < 42189065).$$

At $d_* = 42189065$,

$$F(d_*) = 14342162, \qquad \lfloor \lambda d_* \rfloor = 14342163.$$

Proof. Appendix A gives an exact rational enclosure

$$\frac{L}{10^{30}} < \lambda < \frac{L+1}{10^{30}}, \qquad L = 339949771344778929576702809321.$$

Direct integer comparisons give

$$\frac{846547}{2490212} < q < \frac{14342163}{42189065} < \frac{L}{10^{30}} < \frac{L+1}{10^{30}} < \frac{13495616}{39698853}.$$

The outer fractions have determinant one:

$$13495616 \cdot 2490212 - 846547 \cdot 39698853 = 1.$$

Their mediant is $14342163/42189065$. Every separator between q and λ lies strictly between the outer fractions, so Lemma 4.1 gives denominator at least $2490212 + 39698853 = 42189065$. The mediant separates q and λ, attaining that bound. Finally integer division gives $\lfloor Qd_*/S \rfloor = 14342162$, while both rational enclosure endpoints multiplied by d_* have floor 14342163. □

This theorem is inherited from Paper II [2, Proposition 9.3]; an independent floor-sum derivation is included in Appendix A and replayed by the companion. The theorem concerns the first failure, not permanent disagreement afterward. The two floors agree again at $d_* + 1$. Beyond the first failure, any new domain claim must specify and verify the intended map.

For $d > 0$, $0 \leq F(d) < d$, so every nonnegative integer eventually reaches zero. The restriction to any initial interval of integers is invariant. Thus Theorem 4.2 also implies equality of complete forward orbits for every seed below d_*. Strict descent is the relevant property: F is not a strict contraction for the usual distance on integers, since $F(2) = 0$ and $F(3) = 1$.

5 Exact fibers and depth geometry

Let

$$\mathcal{L}(n) = \left\lceil \frac{Sn}{Q} \right\rceil, \qquad A = 3Q - S = 19849314034334.$$

Theorem 5.1 (Fibers and their exact size law). For every $n \geq 0$,

$$F^{-1}(n) = [\mathcal{L}(n), \mathcal{L}(n+1) - 1] \cap \mathbb{Z}.$$

Their cardinalities are two or three and satisfy

$$a_n = 3 - \left(\left\lfloor \frac{(n+1)A}{Q} \right\rfloor - \left\lfloor \frac{nA}{Q} \right\rfloor \right).$$

Writing $r_n = nS \bmod Q$, the size is two exactly when $0 < r_n \leq A$. The increasing size-two indices are

$$n_k = \left\lceil \frac{kQ}{A} \right\rceil - 1, \qquad k = 1, 2, \dots .$$

The size word has least period $P = Q/2 = 169974885672389$, with $A/2 = 9924657017167$ size-two fibers in a period. Its size-two and size-three natural densities are respectively A/Q and $(S - 2Q)/Q$. Consecutive size-two bases differ by 17 or 18.

Proof. The floor equation is $nS \leq Qd < (n+1)S$. Taking integer endpoints proves the interval formula. Since $S = 3Q - A$,

$$\mathcal{L}(n) = 3n - \lfloor nA/Q \rfloor,$$

and subtraction gives the size formula. As $0 < A < Q$, the difference of floors is zero or one. It is one exactly when $nA \bmod Q \geq Q - A$. If $r_n \neq 0$, then $nA \bmod Q = Q - r_n$, giving $0 < r_n \leq A$; when $r_n = 0$ there is no increment. A floor increment occurs when some positive integer k satisfies $nA/Q < k \leq (n+1)A/Q$, equivalently $n < kQ/A \leq n + 1$. This gives the ceiling formula, including equality endpoints.

Since $\gcd(Q, A) = \gcd(Q, S) = 2$, adding $P = Q/2$ preserves the increments, and exactly $A/2$ increments occur in a period. The reduced fraction $(A/2)/P$ has coprime numerator and denominator. If a shorter period t existed, its number of increments divided by t would equal this density, forcing $P \mid t$. Thus P is least. Periodic counting gives the two densities. Finally $17 < Q/A < 18$ by integer comparison; successive values of $\lceil kQ/A \rceil$ differ by either $\lfloor Q/A \rfloor$ or $\lceil Q/A \rceil$. □

In particular, the rational sequence is a periodic mechanical word. At the remote endpoints $n = P - 1$ and $n = P$, the residues are respectively A and zero, so the fiber sizes are two and three. At $k = A/2$, the size-two formula selects $P - 1$. These exact endpoints detect errors that a short initial scan cannot detect.

Corollary 5.2 (Counting on arbitrary intervals). For $m \geq 0$ and $N \geq 1$, the number of size-two fibers among bases $m, \dots, m+N-1$ is

$$\left\lfloor \frac{(m+N)A}{Q} \right\rfloor - \left\lfloor \frac{mA}{Q} \right\rfloor.$$

Its difference from NA/Q has absolute value less than one. The total number of predecessors of these bases is $\mathcal{L}(m+N) - \mathcal{L}(m)$, differing from NS/Q by less than one.

Proof. Sum the consecutive floor increments in Theorem 5.1; all intermediate terms cancel. The difference from NA/Q is a difference of two fractional parts. The fibers concatenate into one interval, and the analogous ceiling difference gives the second bound. □

Theorem 5.3 (Iterated preimages and exact depth). For $k \geq 0$ and $n \geq 0$,

$$F^{-k}(n) = [\mathcal{L}^{\circ k}(n), \mathcal{L}^{\circ k}(n+1) - 1] \cap \mathbb{Z}.$$

Define

$$B_0 = 0, \qquad B_{k+1} = \left\lfloor \frac{S(B_k+1)-1}{Q} \right\rfloor.$$

Then $F^{-k}(0) = [0, B_k] \cap \mathbb{Z}$. If $h(d) = \min\{k : F^k(d) = 0\}$, its positive depth layers are

$$\{d : h(d) = k\} = [B_{k-1}+1, B_k] \cap \mathbb{Z} \qquad (k \geq 1).$$

Proof. The preimage of an integer interval $[a, b]$ is the union of consecutive fibers, namely $[\mathcal{L}(a), \mathcal{L}(b+1)-1]$. Apply this identity inductively, starting with $[n, n]$. For the root interval its next right endpoint is $\mathcal{L}(B_k+1) - 1 = \lfloor (S(B_k+1)-1)/Q \rfloor$. Since zero is fixed, these root preimages are nested, and subtracting consecutive root intervals gives exact first-hitting layers. □

Thus the sets $F^{-k}(0)$ are nested; the depth layers, not those nested sets, partition the state space. The following values follow by exact integer recurrence.

Depth	Right endpoint B_k	Number in D at this depth
0	0	1
1	2	2
2	8	6
3	26	18
4	79	53
5	235	156
6	694	459
7	2044	1350
8	6015	3971
9	17696	11681
10	52057	15071

The deepest layer in D is exactly $[17697, 32767]$. Also $B_{16} = 33728823$ and $B_{17} = 99217081$. Hence the depth-seventeen layer on the full agreement domain $[0, d_* - 1]$ is $[33728824, 42189064]$. For the rational map on $[0, d_*]$ it includes the additional endpoint 42189065.

There are 11,140 nonempty restricted fibers on D: 10,489 of size three, 650 of size two, and one of size

one. The last unrestricted fiber is $F^{-1}(11139) = \{32767, 32768, 32769\}$, clipped to a singleton by D. If one counts children in the rooted tree, rather than preimages including the self-loop at zero, the root has two children. The corresponding numbers of vertices with zero, one, two, and three children are $21628, 1, 651, 10488$. Their edge count is 32767.

The local arrangement can also be enumerated exactly. Put $V_n = F^{-1}(n) \cap D$ and $a_D(n) = |V_n|$, setting $a_D(n) = 0$ for an empty fiber. For each $0 \le n \le 11139$, list the elements $x_1 < \cdots < x_j$ of V_n and form the ordered signature $(a_D(x_1), \dots, a_D(x_j))$. The complete count is

Ordered child-fiber signature	Number of bases
(0)	1
$(0,0)$	429
$(0,0,0)$	6923
$(2,3)$	12
$(3,2)$	11
$(3,3)$	198
$(2,3,3)$	206
$(3,2,3)$	213
$(3,3,2)$	208
$(3,3,3)$	2938
$(3,3,1)$	1
Total	11140

These counts follow by applying the exact endpoint formula to every nonempty base and its children. A zero entry means that the indicated child has no predecessor inside D; it does not mean that the child lies outside D. This table includes the self-preimage at zero. Removing that loop gives the rooted-tree child counts stated above. Strict descent makes every seed reach the same root and rules out all other cycles.

The first-disagreement seed also has an exact local description:

$$F^{-1}(14342162) = \{42189063, 42189064, 42189065\}, \qquad F^{-1}(14342163) = \{42189066, 42189067\}.$$

Thus d_* is the right endpoint of a three-element fiber immediately before a two-element fiber. This follows by integer evaluation of $\mathcal{L}$; minimal disagreement with the real floor map is a separate consequence of Theorem 4.2. As a small reconstructible example,

$$399 \longmapsto 135 \longmapsto 45 \longmapsto 15 \longmapsto 5 \longmapsto 1 \longmapsto 0.$$

On each fiber, the remainder $\rho(d) = Qd - SF(d)$ increases by Q between successive seeds. When the image increases by one, its change is $Q - S$. These statements follow immediately by subtraction, and explain the local remainder pattern without a statistical assumption.

The intrinsic intervals above should be distinguished from a computational division of D into 32 consecutive blocks of 1024 seeds. Nine such blocks have 348 distinct images and 23 have 349. Their image counts sum to 11,159 because images overlap between blocks; their seed sets do not overlap. This regularity follows from the floor map and does not constitute a group symmetry argument.

6 Recovery and approximation of the marked construction

For bookkeeping at the absorbing register value, define $\Gamma_\kappa(0) = \{\alpha\}$, with every point label collapsed to α. This degenerate record has no positive scale and no defined ratio $(G + C)/a$.

Theorem 6.1 (Geometric recovery from an admissible record). Fix $h > 0$, κ, and the construction rules of Section 2. For a registered scale $a_0 = hd_0$ with $d_0 \in D$, retain its complete ordered record

$$\mathcal{R}(d_0) = (0; \rho_0, \dots, \rho_{k-1})$$

under F, stopped at its first zero. Then $(h, \kappa, \mathcal{R}(d_0))$ determines the entire initial marked construction and every registered intermediate construction $\Gamma_\kappa(hd_j)$ uniquely. Every such record has $k \leq 10$. The same registered sequence is obtained with $\lfloor \lambda d \rfloor$.

Proof. Theorem 3.2 reconstructs each integer d_j exactly. Multiplication by the retained positive scale h gives each geometric scale. Proposition 2.1 and the definitions, including the zero convention, recover each marked construction. Theorem 5.3 gives the depth bound, and Theorem 4.2 gives equality of the two integer maps throughout the invariant domain D. Uniqueness follows at each inverse step. □

This is an exact round trip for the stated family. For a general scalar observation of a general shape, Proposition 2.2 remains the applicable condition. The remainders cannot recover omitted placement or unobserved degrees of freedom. The theorem also does not assert optimal data compression: distinguishing all 32,768 seeds requires at least 15 bits in any fixed-length lossless code, whereas an explicit trajectory record retains additional verification information.

Theorem 6.2 (Geometric effect of quantization). Let $d_0 \in D$ and extend its registered orbit by zeros after termination. Set

$$e_j = \lambda d_j - d_{j+1} = \rho_j/S + (\lambda - q)d_j.$$

Here $\rho_j = 0$ after termination. For every $n \geq 1$,

$$0 \leq \lambda^n d_0 - d_n = \sum_{j=0}^{n-1} \lambda^{n-1-j} e_j < \frac{1-\lambda^n}{1-\lambda}, \qquad 0 \leq e_j < 1.$$

Define

$$R_\Gamma = \sqrt{c_\delta^2 + (\varphi^2 + s_\delta)^2}.$$

Under the natural correspondence of point labels, the greatest distance between the continuously contracted construction $\Gamma_\kappa(h\lambda^n d_0)$ and the registered construction $\Gamma_\kappa(hd_n)$ is exactly

$$R_\Gamma h(\lambda^n d_0 - d_n) < R_\Gamma h \frac{1-\lambda^n}{1-\lambda}.$$

Their Hausdorff distance is exactly the displayed corresponding-point distance.

Proof. Theorem 4.2 gives $d_{j+1} = \lfloor \lambda d_j \rfloor$, hence $e_j \in [0, 1)$. Subtracting the rational quotient identity gives its displayed expression. Expanding $d_{j+1} = \lambda d_j - e_j$ by induction proves the finite remainder sum. Bounding its terms by the geometric series gives the strict upper bound. This scalar argument is the estimate of Paper I [1].

Write each point in the unit-scale outline as $\alpha + w$ with w in the prescribed planar template. Corresponding points at scales a, b differ by $(a - b)w$. The greatest template norm is R_Γ: the circular arc

has norm one, while the straight pieces attain their largest norm at the outer labeled endpoints. Thus the supremum of corresponding-point distances is $R_\Gamma|a-b|$, with equality at those endpoints. For the matching Hausdorff lower bound, assume $a \geq b$ and choose a template point w_* of norm R_Γ. Every point $\alpha + bw$ in the smaller copy satisfies

$$|aw_* - bw| \geq a|w_*| - b|w| \geq (a-b)R_\Gamma.$$

Thus the distance from $\alpha + aw_*$ to the smaller copy is at least $(a-b)R_\Gamma$. The correspondence provides the reverse bound. Substitute $a = h\lambda^n d_0$ and $b = hd_n$. $\square$

The argument proves a useful general statement: for every nonempty compact set $K \subset \mathbb{R}^m$, $R_K = \max_{x\in K}|x|$, center α, and $a, b \geq 0$,

$$d_H(\alpha + aK, \alpha + bK) = R_K|a-b|.$$

Neither convexity nor containment of the origin is needed. At $n = 0$ the discrepancy is zero; the strict geometric-series inequality in Theorem 6.2 is stated for $n \geq 1$. The accumulation identity holds mathematically whether or not a program stores its terms. Retaining the remainders is what permits exact inverse reconstruction. For $0 < \lambda < 1$ the discrepancy stays below $hR_\Gamma/(1-\lambda)$, so repeated rounding does not create an unbounded error in this contracting system.

For the rational continuous scale $q^n d_0$, the same expansion holds with q and ρ_j/S. Iterating a floor is not equivalent to flooring a power: $F(F(26)) = 2$, whereas $\lfloor q^2 \cdot 26 \rfloor = 3$. For a positive initial scale, the real sequence $h\lambda^n d_0$ stays positive at every finite n even after the register has reached zero. The absorbing register state therefore represents quantized loss of scale, not finite-time collapse of the continuous motion.

7 Smooth incompressible realization

This section retains the corrected geometric fluid realization [8] and supplies its proofs. It describes an exact continuous motion to which the observations of the preceding sections can be applied.

7.1 A quadratic form and a volume-preserving family

For $w = (x, y)$, define

$$H = \begin{pmatrix} 1 & 1/2 \\ 1/2 & -1 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad K = \frac{2M - I}{\sqrt{5}} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix},$$
$$\mathcal{N}(w) = w^T H w = x^2 + xy - y^2, \qquad \ell_+(w) = x + \varphi y, \quad \ell_-(w) = x + \varphi' y.$$

Proposition 7.1 (Exact deformation identities). For $c > 0$ and $\eta \in \mathbb{R}$, set

$$\Phi_{c,\eta}(w, z) = (ce^{\eta K}w, c^{-2}z).$$

These maps form a group with $\Phi_{c,\eta}\Phi_{d,\theta} = \Phi_{cd,\eta+\theta}$ and have determinant one. They satisfy

$$\ell_\pm(ce^{\eta K}w) = ce^{\pm\eta}\ell_\pm(w), \qquad \mathcal{N}(ce^{\eta K}w) = c^2\mathcal{N}(w).$$

Consequently $J(w, z) = z\mathcal{N}(w)$ is invariant under every $\Phi_{c,\eta}$.

Proof. Direct multiplication gives $K^2 = I$, $K^T = K$, $\operatorname{tr} K = 0$, and $K^T H + HK = 0$. The row vectors defining $\ell_\pm$ have eigenvalues ± 1 for right multiplication by K, so the exponential gives the two observation formulas. Also

$$\mathcal{N}(w) = (x + \varphi y)(x + \varphi' y),$$

because $\varphi + \varphi' = 1$ and $\varphi\varphi' = -1$. Multiplying the observation formulas proves the norm identity. The exponentials commute, giving the composition law; their determinant is $\exp(\eta \operatorname{tr} K) = 1$. The planar determinant c^2 is canceled by the axial factor c^{-2}. Multiplying the transformed axial coordinate and quadratic form proves invariance of J. □

For $\eta = 0$, the marked plane undergoes a similarity $w \mapsto cw$. Thus the whole marked construction has scale ac and retains $(G + C)/a = \Omega_0$ when measured at its current scale. For any fixed real linear observation L and $V = 1 + L(w)$,

$$V(\Phi_{c,0}(w, z)) = 1 + c(V(w, z) - 1).$$

At $c = \lambda$, this is exactly T. For $\eta \neq 0$, the map is generally anisotropic and circles become ellipses; the same Euclidean shape statement does not apply. The common family does not identify these different motions.

Proposition 7.2 (The matrix observation). Put $A_\Omega = \tau M$. Then

$$\ell_+(A_\Omega w) = \lambda \ell_+(w), \quad \mathcal{N}(A_\Omega w) = -\tau^2 \mathcal{N}(w), \quad M^2 = e^{2(\log\varphi)K}.$$

For a real schedule $\sigma(t)$, take $c(t) = \tau^{2\sigma(t)}$ and $\eta(t) = 2\sigma(t)\log\varphi$. The relative motion from a reference time t_0 satisfies

$$\ell_+(w(t)) = \lambda^{2(\sigma(t) - \sigma(t_0))} \ell_+(w(t_0)).$$

When $\sigma(t) - \sigma(t_0) = 1$, its full map is

$$\Phi_{c(t)/c(t_0),\, \eta(t) - \eta(t_0)}(w, z) = (A_\Omega^2 w, \tau^{-4} z).$$

The simpler exponent $2\sigma(t)$ applies when $\sigma(t_0) = 0$.

Proof. The eigenvalues of M are φ and $\varphi' = -\varphi^{-1}$, and the same eigenvectors have K eigenvalues $1, -1$. Squaring the eigenvalues gives φ^2, φ^{-2}, which are those of the displayed exponential. Multiplication by τ gives the first observation identity. The two linear factors of $\mathcal{N}$ are multiplied by $\tau\varphi$ and $\tau\varphi'$, whose product is $-\tau^2$. Finally $c(t)e^{\eta(t)} = \lambda^{2\sigma(t)}$. Dividing this multiplier by its value at t_0 gives the relative observation and the stated two-step map. □

The observation $w \mapsto \ell_+(w)$ is not injective. It gives a semiconjugacy to scalar multiplication, not a reversible representation of the plane. Moreover $\det A_\Omega = -\tau^2 < 0$, so a plane-invariant smooth flow starting from the identity cannot realize a single A_Ω as its planar flow map: the planar Jacobian determinant stays positive by its variational equation. The displayed realization correctly uses A_Ω^2.

7.2 Affine solutions and localization

The incompressible equation is

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f, \qquad \nabla \cdot u = 0, \qquad \nu > 0.$$

Theorem 7.3 (Affine realization). Let $c(t) > 0$ and $\eta(t)$ be smooth. Put $\gamma = \dot{c}/c$, $\beta = \dot{\eta}$, and

$$B(t) = \begin{pmatrix} \gamma I_2 + \beta K & 0 \\ 0 & -2\gamma \end{pmatrix}.$$

Then

$$u_{\mathrm{aff}}(X,t) = B(t)X, \qquad p_{\mathrm{aff}}(X,t) = -\tfrac{1}{2}X^T(\dot{B} + B^2)X$$

solve the equation with $f = 0$ for every $\nu > 0$. Their relative particle map is

$$\Psi_{t,t_0} = \Phi_{c(t)/c(t_0),\, \eta(t)-\eta(t_0)}.$$

Proof. The matrix B is symmetric and traceless. Hence $\nabla \cdot (BX) = 0$, $\Delta(BX) = 0$, and

$$\partial_t(BX) + (BX \cdot \nabla)(BX) = (\dot{B} + B^2)X.$$

The symmetric matrix $\dot{B} + B^2$ gives precisely its negative as the pressure gradient. Differentiating the proposed relative map gives $\partial_t \Psi = B\Psi$ and $\Psi_{t_0,t_0} = I$. Uniqueness for this linear ODE proves the flow formula. □

Except at times when $B = 0$, this affine field has infinite kinetic energy on $\mathbb{R}^3$: its square grows quadratically on an open cone on which $BX \neq 0$. The following theorem supplies the finite-energy realization actually used in the geometric interface.

Theorem 7.4 (Finite-energy realization of a prescribed finite motion). Let $T_0 > 0$, let $c, \eta \in C^\infty(\mathbb{R})$ with $c > 0$, and assume $c = 1$, $\eta = 0$ near and before zero and that both functions are constant near and after T_0. For every compact marked set $\mathcal{K} \subset \mathbb{R}^3$, there are smooth, compactly supported velocity and pressure fields u, p, independent of ν, such that for each $\nu > 0$ there is a smooth compactly supported force f_ν with the following properties. The equation holds globally; the fluid starts at rest; its kinetic energy is uniformly finite; and each $X_0 \in \mathcal{K}$ follows exactly $\Phi_{c(t),\eta(t)}(X_0)$. The force vanishes on an open neighborhood of all these marked trajectories.

Proof. The complete trajectory set

$$\mathcal{T} = \{\Phi_{c(t),\eta(t)}(X_0) : X_0 \in \mathcal{K},\ 0 \le t \le T_0\}$$

is compact. Choose $\chi \in C_c^\infty(\mathbb{R}^3)$, equal to one on a neighborhood of $\mathcal{T}$. Such a choice can be made explicitly. Put $b(s) = e^{-1/s}$ for $s > 0$ and $b(s) = 0$ otherwise, and

$$H_0(s) = \frac{b(s)}{b(s) + b(1-s)}.$$

Every derivative of b at zero vanishes, since powers of $1/s$ times $e^{-1/s}$ tend to zero. Thus H_0 is smooth, zero for $s \le 0$, and one for $s \ge 1$. If $\mathcal{T}$ lies strictly inside a ball of radius $R_0 < R_1$, take

$$\chi(X) = 1 - H_0\left(\frac{|X|^2 - R_0^2}{R_1^2 - R_0^2}\right).$$

Define the vector potential, velocity, and pressure by

$$\mathcal{P}(X,t) = \tfrac{1}{3}(BX \times X), \qquad u = \nabla \times (\chi \mathcal{P}), \qquad p = \chi p_{\text{aff}}.$$

For a traceless linear field, the curl identity gives

$$\nabla \times (BX \times X) = 3BX,$$

because the four terms are $3BX$, $-X \operatorname{tr} B = 0$, BX, and $-BX$. Thus $u = BX$ wherever $\chi = 1$, while $\nabla \cdot u = 0$ everywhere. Set

$$f_\nu = \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p.$$

This enforces the momentum equation. All fields are smooth with compact spatial support. Since B and its derivatives vanish near the temporal endpoints and outside the prescribed motion interval, the fields have support in a compact time subinterval of $(0, T_0)$. Theorem 7.3 gives $f_\nu = 0$ where $\chi = 1$; it also vanishes where χ is identically zero. Its support is therefore confined to the transition region surrounding the marked motion.

Smoothness and compact support give uniform bounds on u and its spatial derivatives, and hence a global unique particle flow and a uniform finite-energy bound. Each prescribed marked curve remains where $u = BX$ and solves this particle ODE. Uniqueness identifies it with the actual fluid trajectory. □

Corollary 7.5 (Work and viscosity). For these localized fields,

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^3} |u|^2\, dX + \nu \int_{\mathbb{R}^3} |\nabla u|^2\, dX = \int_{\mathbb{R}^3} f_\nu \cdot u\, dX.$$

If u is not identically zero, then

$$\int_0^{T_0} \left(\int_{\mathbb{R}^3} f_\nu \cdot u\, dX \right) dt = \nu \int_0^{T_0} \left(\int_{\mathbb{R}^3} |\nabla u|^2\, dX \right) dt > 0.$$

The same nonzero compactly supported velocity cannot solve the equation with the same force at two distinct viscosities, even if the pressures may differ.

Proof. Dot the equation with u and integrate. Compact support removes boundary terms; divergence zero eliminates pressure work and the advection integral. Integration by parts gives the negative squared gradient from the Laplacian. The velocity vanishes at both temporal endpoints, proving the integrated equality. A zero gradient integral would make u spatially constant at every time and therefore zero by compact support. For the last assertion, subtract the two equations, dot with u, and integrate in space. The pressure difference again contributes zero and gives $(\nu_2 - \nu_1) \int |\nabla u|^2 = 0$, forcing $u = 0$. □

The affine core is irrotational, but its gradient need not vanish: $|\nabla u_{\text{aff}}|^2 = 6\gamma^2 + 2\beta^2$. Thus $\Delta u_{\text{aff}} = 0$ does not mean zero strain or zero viscous dissipation. A nonzero smooth compactly supported divergence-free field must have nonzero curl somewhere: otherwise the vector Laplacian identity would give $\Delta u = 0$, and integration by parts would force $u = 0$. The localization therefore creates vorticity outside the irrotational core.

The cubic invariant is guaranteed on marked trajectories, not on every trajectory in the transition

region. Indeed, the triple-product identity gives

$$u = \left(\chi + \frac{X \cdot \nabla\chi}{3}\right) BX - \frac{BX \cdot \nabla\chi}{3} X.$$

Using $(BX) \cdot \nabla J = 0$ and Euler's homogeneous identity $X \cdot \nabla J = 3J$ yields $u \cdot \nabla J = -J(BX \cdot \nabla\chi)$, which can be nonzero. Also $J = 0$ on the whole plane $z = 0$, so it cannot identify the particular marked curve there.

Rigid placements are handled without changing the theorem. Extend u, v to the rotation $R = (u, v, u \times v)$ and replace X by $R^T(X - \alpha)$, the velocity and force by their R-rotates, and pressure by composition. The chain rule preserves every equation. This realizes the arbitrary placement in Section 2. The positive-scale and finite-duration assumptions exclude a singular collapse endpoint.

7.3 Extension to higher ambient dimension

The planar generator admits a direct extension to every finite dimension $m \geq 3$. Write $X = (w, z)$ with $w \in \mathbb{R}^2$ and $z \in \mathbb{R}^{m-2}$. The extra coordinates compensate the planar area change; they are mathematical ambient coordinates, with no claim that an m-dimensional fluid is a physical medium.

Theorem 7.6 (Volume compensation and localization in dimension m). Define

$$\Phi^{(m)}_{c,\eta}(w, z) = \left(ce^{\eta K} w, c^{-2/(m-2)} z\right), \qquad c > 0.$$

This family has determinant one, has the same multiplicative and additive composition law as Proposition 7.1, and preserves the homogeneous polynomial

$$J_m(X) = \mathcal{N}(w) \prod_{j=1}^{m-2} z_j.$$

For smooth c, η its instantaneous generator is the symmetric traceless matrix

$$B_m = \operatorname{diag}\left(\gamma I_2 + \beta K, -\frac{2\gamma}{m-2} I_{m-2}\right), \qquad \gamma = \dot{c}/c, \quad \beta = \dot{\eta}.$$

With the schedule and compact-set hypotheses of Theorem 7.4, this motion has a smooth, compactly supported, finite-energy forced incompressible realization on $\mathbb{R}^m$. For an arbitrary smooth traceless linear field $v(X, t) = B(t)X$, one such divergence-free localization is

$$\begin{aligned} \mathcal{A}_{ij}(X, t) &= \frac{v_i X_j - v_j X_i}{m}, \\ u_i(X, t) &= \sum_{j=1}^{m} \partial_j(\chi \mathcal{A}_{ij}), \\ u &= \left(\chi + \frac{X \cdot \nabla\chi}{m}\right) BX - \frac{BX \cdot \nabla\chi}{m} X. \end{aligned}$$

Proof. The planar determinant is c^2 because $\operatorname{tr} K = 0$. The transverse determinant is $(c^{-2/(m-2)})^{m-2} = c^{-2}$, proving volume preservation. The composition law follows componentwise. The quadratic factor in J_m is multiplied by c^2, and the product of transverse coordinates by c^{-2}, proving its invariance. Differentiation gives B_m, whose trace is $2\gamma - 2\gamma = 0$.

For the localization, $\mathcal{A}_{ij} = -\mathcal{A}_{ji}$. Commutativity of mixed derivatives and cancellation in pairs therefore give

$$\nabla \cdot u = \sum_{i,j} \partial_i \partial_j (\chi \mathcal{A}_{ij}) = 0.$$

Since v is linear, Euler's identity gives $\sum_j X_j \partial_j v_i = v_i$. Consequently

$$\sum_j \partial_j (v_i X_j) = (m+1) v_i, \qquad \sum_j \partial_j (v_j X_i) = (\operatorname{tr} B) X_i + v_i = v_i,$$

so $\sum_j \partial_j \mathcal{A}_{ij} = v_i$. The product rule yields the displayed expression for u, which equals BX wherever $\chi = 1$. For $B = B_m$, put

$$p = -\frac{1}{2} \chi X^T (\dot{B} + B^2) X, \qquad f_\nu = \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p.$$

The matrix $\dot{B} + B^2$ is symmetric. The affine momentum calculation of Theorem 7.3 therefore applies in m dimensions. Choose χ equal to one on a neighborhood of the compact trajectory set, using the same function of $|X|^2$ as in Theorem 7.4. Every field is smooth with compact support, the force vanishes near the marked trajectories, and ODE uniqueness identifies those trajectories. The support and smoothness arguments for finite energy, the rest endpoints, and the energy identity are dimension independent. □

For this generator,

$$\operatorname{tr}(B_m^T B_m) = \frac{2m}{m-2} \gamma^2 + 2\beta^2.$$

This follows by expanding the two planar eigenvalues $\gamma \pm \beta$ and the $m-2$ equal transverse eigenvalues. The polynomial J_m has degree m and satisfies $(BX) \cdot \nabla J_m = 0$ for the affine core. The localized field obeys

$$u \cdot \nabla J_m = -J_m (BX \cdot \nabla \chi),$$

by Euler's identity $X \cdot \nabla J_m = m J_m$. Thus the invariant is exact on the selected core trajectories and need not survive motion through the cutoff region.

The extra dimensions do not remove the volume constraint. A uniform contraction in all m directions has determinant less than one and cannot itself be an incompressible flow map. Here contraction of the marked plane is accompanied by transverse expansion. The theorem covers arbitrary finite positive schedules; it excludes a singular endpoint $c = 0$. A coordinate projection from $\mathbb{R}^m$ to three displayed coordinates is a visualization and generally loses information about the higher-dimensional state.

8 Finite certificates with continuous geometric meaning

8.1 Polynomial integration from a finite basis

Let σ be probability area measure on the unit sphere S^2. For nodes $\omega_i \in S^2$ and weights $w_i \in \mathbb{R}$, set $\mathcal{Q}(p) = \sum_i w_i p(\omega_i)$.

Theorem 8.1 (Finite moment certificate). Fix a nonnegative integer t. If

$$\mathcal{Q}(x^a y^b z^c) = \int_{S^2} x^a y^b z^c \, d\sigma \qquad (a, b, c \geq 0,\ a + b + c \leq t),$$

then $\mathcal{Q}(p) = \int p \, d\sigma$ for every real polynomial of total degree at most t. These finitely many identities are also necessary. The right-hand moment is zero if any exponent is odd. For even exponents it is

$$\int x^{2a} y^{2b} z^{2c} \, d\sigma = \frac{(2a-1)!!(2b-1)!!(2c-1)!!}{(2a+2b+2c+1)!!},$$

with $(-1)!! = 1$.

Proof. The monomials span the polynomial space over $\mathbb{R}$, and both functionals are real linear; this proves sufficiency and necessity. Reflection in an odd coordinate reverses the integrand and preserves measure, giving zero. For the even formula, integrate $e^{-|X|^2} x^{2a} y^{2b} z^{2c}$ over $\mathbb{R}^3$ in two ways. Product integration gives $\Gamma(a+1/2)\Gamma(b+1/2)\Gamma(c+1/2)$. Polar integration gives $4\pi \int_{S^2} x^{2a} y^{2b} z^{2c} \, d\sigma$ multiplied by $\frac{1}{2}\Gamma(a+b+c+3/2)$. Substitute $\Gamma(n+1/2) = (2n-1)!!\sqrt{\pi}/2^n$ and simplify. □

For an orthogonal finite group G, a G-invariant node rule satisfies $\mathcal{Q}(p) = \mathcal{Q}(\overline{p})$, where $\overline{p} = |G|^{-1} \sum_{g \in G} p \circ g$. The sphere integral has the same property. Thus checking a spanning set for the invariant polynomials suffices. This follows by averaging and Theorem 8.1; any smaller proposed spanning set needs its own spanning proof. Paper VI uses this reduction to obtain higher-degree positive rules [6].

Theorem 8.2 (A complete small exact certificate). Let V consist of the twelve vertices

$$(0, \pm 1, \pm\varphi), \qquad (\pm 1, \pm\varphi, 0), \qquad (\pm\varphi, 0, \pm 1),$$

with signs chosen independently. The nodes $\omega_v = v/\sqrt{\varphi + 2}$, each of weight $1/12$, integrate every polynomial of degree at most five exactly. They do not integrate all polynomials of degree six exactly.

Proof. Every vertex has squared norm $\varphi + 2$. Coordinate sign changes preserve the node set, so any monomial having an odd exponent has node average zero. Cyclic permutation preserves the set. Among nonconstant even monomials of degree at most five, only the types x^2, x^4, and x^2y^2 remain. Their averages are

$$\begin{aligned} \mathcal{Q}(x^2) &= \frac{1+\varphi^2}{3(\varphi+2)} = \frac{1}{3}, \\ \mathcal{Q}(x^4) &= \frac{1+\varphi^4}{3(\varphi+2)^2} = \frac{1}{5}, \\ \mathcal{Q}(x^2y^2) &= \frac{\varphi^2}{3(\varphi+2)^2} = \frac{1}{15}. \end{aligned}$$

The identities use $\varphi^2 = \varphi + 1$ and $(\varphi+2)^2 = 5\varphi^2$. The constant moment is one. Cyclic permutation carries x^2y^2 through all three unordered coordinate pairs, so the listed types exhaust the even monomials in this degree range. These are precisely the moments in Theorem 8.1, proving degree-five exactness. At degree six,

$$\mathcal{Q}(x^6) = \frac{1+\varphi^6}{3(\varphi+2)^3} = \frac{2}{15} \neq \frac{1}{7} = \int x^6 \, d\sigma.$$

This supplies a concrete failure witness, with error $-1/105$. It disproves exactness on the entire degree-six polynomial space; particular higher-degree polynomials can still be integrated exactly. □

The companion additionally checks all $\binom{8}{3} = 56$ monomials individually. Even-degree normalization lies in $\mathbb{Q}(\sqrt{5})$; odd-degree raw sums vanish exactly before any square-root normalization is needed. This is the classical icosahedral spherical 5-design [15,16], used here with its complete elementary proof and an exact reproducible certificate.

Theorem 8.3 (Universal fluid observables from the finite certificate). Any degree-two exact probability rule on S^2 satisfies, for every real 3×3 matrix B and every $r \geq 0$,

$$\sum_i w_i |Br\omega_i|^2 = \frac{r^2}{3} \operatorname{tr}(B^T B),$$

$$\sum_i w_i (Br\omega_i)(Br\omega_i)^T = \frac{r^2}{3} BB^T.$$

The sums equal the corresponding spherical integrals. In particular, the kinetic energy of $u(X) = BX$ inside the radius-R ball is

$$\frac{1}{2} \int_{|X| \leq R} |BX|^2 \, dX = \frac{2\pi R^5}{15} \operatorname{tr}(B^T B).$$

Proof. Degree-two exactness gives $\sum_i w_i \omega_i \omega_i^T = I/3$ by the second moments in Theorem 8.1. Multiplication on the two sides by rB and rB^T proves the matrix formula; taking the trace proves the scalar formula. They equal the integrals because the same moment matrix is the sphere integral. Polar integration then multiplies $r^2 \operatorname{tr}(B^T B)/3$ by $4\pi r^2 dr$, and the kinetic-energy factor $1/2$ gives the stated result. □

Here the input certificate is finite, but its interpretation quantifies over every real B and r. The proof by linearity and matrix multiplication supplies that extension. Applied to Section 7, $\operatorname{tr}(B^T B) = 6\gamma^2 + 2\beta^2$. The ball formula concerns an affine core where the cutoff is one; the transition region contributes additional energy. The matrix observable is the velocity second moment $\mathbb{E}[uu^T]$, also used as a kinetic stress. It is not the full Cauchy stress, which involves pressure and velocity derivatives.

One related quadratic observation has the same exact status. If a random planar vector W has finite second moment, mean $\bar{W}$, and covariance matrix R, then

$$\mathbb{E}\mathcal{N}(W) - \mathcal{N}(\bar{W}) = \operatorname{tr}(HR), \qquad |\operatorname{tr}(HR)| \leq \frac{\sqrt{5}}{2} \operatorname{tr} R.$$

To prove this, expand $W = \bar{W} + (W - \bar{W})$ and use zero mean of the fluctuation. Diagonalize H, whose eigenvalues are $\pm\sqrt{5}/2$; positivity of R bounds the trace by the largest absolute eigenvalue times $\operatorname{tr} R$. The observation is not a complete description of covariance: any isotropic covariance has zero observed trace because $\operatorname{tr} H = 0$.

8.2 Bounds on a whole region from finitely many coefficients

Theorem 8.4 (A finite certificate for a rational-function integral). Let a triangle Δ be partitioned by repeated midpoint subdivision into finitely many subtriangles. On each cell, express polynomials N, D in a fixed-degree barycentric Bernstein basis. Suppose the coefficient bounds are $n_- \leq n_+$ for N and $0 < d_- \leq d_+$ for D. Then throughout that cell,

$$\ell \leq N/D \leq u,$$

where ℓ and u are respectively the minimum and maximum of the four ratios formed from n_-, n_+ and d_-, d_+. Therefore

$$\sum_C |C|\ell_C \le \int_\Delta N/D \le \sum_C |C|u_C.$$

If the left sum is strictly positive, the integral is positive. All these conclusions follow from finite exact coefficient and sign checks when the coefficients and cell areas lie in an explicitly ordered exact number field.

Proof. The Bernstein functions on a triangle are

$$b_{ijk}(a,b,c) = \frac{m!}{i!j!k!}a^i b^j c^k, \qquad i+j+k=m, \quad a,b,c \ge 0, \quad a+b+c=1.$$

They are nonnegative and sum to $(a+b+c)^m = 1$. Thus a polynomial expressed in this basis lies between its least and greatest coefficient on the entire cell. The denominator lower bound is strictly positive. On a rectangle with positive denominator coordinate, the extrema of the ratio of the two coordinates occur among its four corners, proving the quotient enclosure even when numerator bounds cross zero. Integrate the cellwise inequalities and sum. Midpoint subdivision covers the parent triangle by four triangles with disjoint interiors; induction proves coverage at every depth, and cell boundaries have area zero. □

The supplied Paper IV material describes an application on $4^6 = 4096$ cells, with coefficients in $\mathbb{Q}(\sqrt{5})$ [4, Section 8 and Appendix A]. Such a certificate bounds an integral; it does not require the integrand to be pointwise positive. The present theorem establishes the implication from valid coefficients and coverage to the integral bounds. The 4096-cell payload is not a premise of the results here and is not included in the present execution record.

8.3 Higher-dimensional moments and localized energy

Proposition 8.5 (Sphere moments in arbitrary dimension). Let σ_{m-1} be probability area measure on $S^{m-1} \subset \mathbb{R}^m$, where $m \ge 2$. A monomial with any odd exponent has zero integral. If $a_1, \dots, a_m$ are nonnegative integers and $A = \sum_i a_i$, then

$$\int_{S^{m-1}} \prod_{i=1}^m \omega_i^{2a_i}\, d\sigma_{m-1} = \frac{\prod_i (2a_i - 1)!!}{\prod_{j=0}^{A-1}(m+2j)}.$$

The denominator is one when $A = 0$. A finite rule matching all these moments through degree t integrates every real polynomial of degree at most t.

Proof. Coordinate reflections prove the odd case. Product integration of $e^{-|X|^2}\prod_i X_i^{2a_i}$ gives $\prod_i \Gamma(a_i + 1/2)$. Polar integration gives the sphere area $2\pi^{m/2}/\Gamma(m/2)$, the unknown normalized moment, and the radial factor $\Gamma(A + m/2)/2$. Equating them and applying the gamma recurrence gives the formula. The finite-basis implication follows by linearity exactly as in Theorem 8.1. □

This supplies exact target moments in every finite dimension. It does not supply higher-dimensional node sets by projecting or extending the twelve three-dimensional nodes. Any proposed rule must satisfy its own moment identities.

Theorem 8.6 (Angular energy and the localization obstruction). Fix a real symmetric traceless $m \times m$ matrix B, $m \geq 3$. Let the smooth cutoff be radial, $\chi(X) = b(r)$ with $r = |X|$, and put

$$t(r) = \frac{rb'(r)}{m}, \qquad a(r) = b(r) + t(r), \qquad q_B(\omega) = \omega^T B\omega.$$

At $X = r\omega$ the localized field of Theorem 7.6 is

$$u(r\omega) = r\{a(r)B\omega - t(r)q_B(\omega)\omega\}.$$

For each fixed r, its squared speed on the sphere is a polynomial of degree at most four after using $|\omega|^2 = 1$. Its exact angular average is

$$\int_{S^{m-1}} |u(r\omega)|^2 \, d\sigma_{m-1} = r^2 \operatorname{tr}(B^2) \left\{ \frac{b(r)^2}{m} + \frac{2b(r)t(r) + t(r)^2}{m+2} \right\}.$$

In three dimensions, the twelve-node rule of Theorem 8.2 therefore gives the exact angular kinetic-energy density throughout both the affine core and the cutoff region. In contrast, individual entries of uu^T can require degree six. For $m = 3$, $B = \operatorname{diag}(1, 1, -2)$, and the stated orientation of the twelve-node rule,

$$\mathcal{Q}(u_z^2) - \int_{S^2} u_z^2 \, d\sigma = -\frac{3}{35} r^2 t(r)^2.$$

Proof. Substitute $\nabla\chi = b'(r)\omega$ into Theorem 7.6. For $r > 0$ this gives the field formula, which extends to $r = 0$ by continuity. Taking its squared norm and using $|\omega|^2 = 1$ yields

$$|u|^2 = r^2 \left\{ a^2 \omega^T B^2 \omega + (t^2 - 2at) q_B(\omega)^2 \right\}.$$

Proposition 8.5 gives

$$\int \omega^T B^2 \omega \, d\sigma_{m-1} = \frac{\operatorname{tr}(B^2)}{m}, \qquad \int q_B^2 \, d\sigma_{m-1} = \frac{(\operatorname{tr} B)^2 + 2\operatorname{tr}(B^2)}{m(m+2)}.$$

For completeness, orthogonally diagonalize B. Then $q_B = \sum_i \mu_i \omega_i^2$; the moments $\int \omega_i^4 = 3/[m(m+2)]$ and $\int \omega_i^2 \omega_j^2 = 1/[m(m+2)]$ give the second identity by expansion. Use $\operatorname{tr} B = 0$ and $a = b + t$ to simplify to the asserted energy formula. Degree-four exactness suffices because the displayed squared-speed expression has that degree on the sphere.

For the counterexample, write the third spherical coordinate as z. Since $q_B = 1 - 3z^2$ on S^2,

$$u_z/r = -(2a + t)z + 3tz^3.$$

Its square has terms in z^2, z^4, z^6. The first two are integrated exactly; the coefficient of z^6 is $9t^2$. Theorem 8.2 gives degree-six error $-1/105$ in each coordinate by cyclic symmetry, yielding $-9r^2t^2/105$. □

The distinction is observable dependent. The trace $|u|^2$ has a lower-degree expression on the sphere because $|\omega|^2 = 1$, while its individual matrix components need not. Total kinetic energy in three

dimensions still requires radial integration:

$$E = \frac{4\pi}{2} \int_0^\infty r^2 \left(\int_{S^2} |u(r\omega)|^2 \, d\sigma \right) dr.$$

Angular exactness alone does not certify that remaining one-dimensional integral, every derivative of the field, or every nonlinear observable of the forcing. The existence proof for the localized fluid uses Theorem 7.4, independently of the availability of a cubature rule.

9 Compatible algebraic and coordinate descriptions

The interfaces in this section locate the remaining algebraic and spiral observations without treating shared constants as a proof of shared geometry.

9.1 Scalar fields and geometric coordinates

Proposition 9.1 (Reversible scalar package and a field boundary). The scalar packages satisfy

$$\varphi = \lambda/\tau, \qquad \pi = (1-\lambda)/\tau, \qquad \lambda^2 - \tau\lambda - \tau^2 = 0,$$

and

$$\mathbb{Q}(\tau, \lambda) = \mathbb{Q}(\pi, \sqrt{5}).$$

For the six normalized labeled points and the fixed endpoint and frame coefficients in the canonical placement, put

$$K_{\text{coord}} = \mathbb{Q}(\sqrt{5}, \sin\delta, \cos\delta), \qquad K_\Omega = \mathbb{Q}(\pi, \sqrt{5}).$$

Then $K_{\text{coord}} \cap K_\Omega = \mathbb{Q}(\sqrt{5})$.

Proof. The first two identities follow directly from the definitions and show both inclusions of the field equality. Multiplying $\varphi^2 - \varphi - 1 = 0$ by τ^2 gives the relation. Since $e^{i\delta}$ is a root of unity of order 2,592,000, both $\sin\delta$ and $\cos\delta$ are algebraic, so K_{coord} is algebraic over $\mathbb{Q}$. The only elements of $\mathbb{Q}(\sqrt{5})(\pi)$ algebraic over $\mathbb{Q}(\sqrt{5})$ are its constants. Indeed, if a nonconstant rational function $R(\pi)$ were algebraic over that field, the equation expressing $R(\pi)$ as a quotient would make π algebraic over the algebraic extension generated by $R(\pi)$, contradicting the transcendence of π. Both fields contain $\mathbb{Q}(\sqrt{5})$, so their intersection is exactly that field. □

The scalar relations and field normal forms are developed in Paper V [5]. The field intersection concerns the finite marked-coordinate data just specified. It does not assert that every point on the continuous circular arc has coordinates in this algebraic field: the arc contains uncountably many points, whereas an algebraic number field is countable. Arbitrary real placement or scale data can also enlarge the field. Including the selected arc length π changes the field and changes the intersection statement. An exact coefficient $a + b\sqrt{5}$ may be written as

$$a + b\sqrt{5} = (a - b) + (2b/\tau)\lambda.$$

Substituting this identity preserves arithmetic and equalities in cubature certificates. It does not construct the cubature nodes from the marked outline. Nor do the two scalars τ, λ recover its placement and scale; Proposition 2.1 specifies the missing data.

9.2 A continuous spiral observation

Proposition 9.2 (Explicit coordinate transport). On $\mathbb{C}$, define

$$W(z) = e^{-2\pi i|z|}z, \qquad Z(t) = \sqrt{t}\, e^{2\pi i\sqrt{t}} \quad (t \geq 0).$$

The map W is a homeomorphism with inverse $W^{-1}(z) = e^{2\pi i|z|}z$. For $c > 0$,

$$R_c = W^{-1}(cI)W, \qquad R_c(z) = ce^{2\pi i(c-1)|z|}z, \qquad R_cZ(t) = Z(c^2t).$$

For $g(k) = k^2 - k - 1$ on $k \geq \varphi$, define

$$K_c(k) = \frac{1}{2} + \sqrt{c^2g(k) + \frac{5}{4}}.$$

Then $g(K_c(k)) = c^2g(k)$ and

$$R_cZ(g(k)) = Z(g(K_c(k))).$$

Proof. The two formulas for W, W^{-1} preserve modulus, so their phases cancel on composition; they are continuous also at zero. Substitution gives R_c, and $W(Z(t)) = \sqrt{t}$ proves its effect on the spiral. On $[\varphi, \infty)$, the polynomial g increases continuously from zero to infinity and has inverse $g^{-1}(t) = 1/2 + \sqrt{t + 5/4}$. Thus $K_c = g^{-1}(c^2g)$, proving the remaining identities. □

This is an exact transport on a continuous spiral, as developed in the golden-norm dossier [9]. It generally does not preserve integer labels t. Its scaling factor on labels is c^2, not c, because $|Z(t)|^2 = t$. It is a conjugate of Euclidean scaling in the specified coordinates, not in general a Euclidean similarity in the original plane. A discrete ledger for an integer k can reconstruct $g(k)$ by substitution after recovering k; this does not turn the floor map on k into the continuous map K_λ.

9.3 Recursive structures and their mathematical status

Three recursive constructions are available, with different meanings. The predecessor intervals of Section 5 form the intrinsic recursion of the integer map. The family of homothetic outlines records repeated geometric scaling. A multiple-map iterated function system is a further geometric definition, requiring its own maps and hypotheses.

Proposition 9.3 (Repeated scaling and compact recursive limits). Let $K \subset \mathbb{R}^m$ be nonempty and compact, let $\alpha \in \mathbb{R}^m$, and let $0 < r < 1$.

(a) The single similarity $f(X) = \alpha + r(X - \alpha)$ has $\{\alpha\}$ as its unique nonempty compact invariant set.

(b) The set

$$\mathcal{R} = \{\alpha\} \cup \bigcup_{n=0}^{\infty} \{\alpha + r^n x : x \in K\}$$

is compact. If K is the nondegenerate marked outline centered at the origin, then $\dim_H \mathcal{R} = 1$.

(c) Given finitely many similarities $f_i(X) = r_iO_iX + t_i$, with O_i orthogonal and $0 < r_i < 1$, there is a unique nonempty compact set A satisfying $A = \bigcup_i f_i(A)$. If the open set condition holds, its Hausdorff dimension is the unique $s \geq 0$ satisfying $\sum_i r_i^s = 1$.

Proof. For (a), an invariant compact set has maximum distance R from α satisfying $R = rR$, so $R = 0$.

The singleton is invariant. For (b), boundedness follows from boundedness of K. In a convergent sequence of points from the displayed copies, an unbounded subsequence of copy indices converges to α; a bounded subsequence lies in a finite union of compact copies. Thus $\mathcal{R}$ is closed and bounded, hence compact. The marked outline is a finite union of smooth compact arcs and nondegenerate line segments, so it has Hausdorff dimension one. Similarities preserve dimension, and the Hausdorff dimension of a countable union is the supremum of the dimensions of its members. This proves the dimension assertion.

For existence and uniqueness in (c), let $\mathcal{K}$ be the complete metric space of nonempty compact subsets of $\mathbb{R}^m$ with Hausdorff distance. The union operator $\mathcal{F}(C) = \bigcup_i f_i(C)$ satisfies

$$d_H(\mathcal{F}(C), \mathcal{F}(D)) \leq \max_i r_i \, d_H(C, D).$$

The estimate follows by pairing a point in $f_i(C)$ with its image of a nearest point in D, and conversely. The contraction mapping theorem gives the unique compact fixed set. The dimension conclusion is the classical similarity-dimension theorem under the open set condition [17]: there must exist a nonempty bounded open set U such that the sets $f_i(U)$ are pairwise disjoint and lie in U. Its separation hypothesis is part of the conclusion's premises. □

Part (c) records a classical theorem and its precise applicability, rather than assigning that dimension to an image generated by software. The optional sampler in the companion uses $f_i(X) = rX + (1-r)c_i$ for user-supplied centers c_i. Its points illustrate that separately declared construction; they do not prove separation, coverage, or dimension. Repeated copies of the given outline, by themselves, have the dimension in (b) and do not create a new noninteger-dimensional attractor. Likewise a three-dimensional rendering of the integer predecessor tree represents an abstract rooted graph, not a uniquely prescribed spatial geometry.

10 Correct composition and the assembled theorem

10.1 Maps with explicit source data and preconditions

Let a data family X_i be specified by a mathematical membership predicate. An operation e has a source family X_e, a context family C_e, a precondition $P_e(x, c)$, and a defined mathematical output $f_e(x, c)$ in a target family Y_e. A context supplies additional input; it is not an assertion that a missing condition has been proved. Examples include choosing a seed, supplying a frame and resolution, choosing the iteration depth of a preimage, or prescribing the rates of an instantaneous field.

An implementation is locally correct for an exact operation when, on every accepted source and context, its output represents $f_e(x, c)$ and belongs to Y_e. When a proposed output is submitted, acceptance must check its equality with that output, as well as the membership and precondition checks. Matching type names alone is insufficient.

Theorem 10.1 (Composition of locally correct interfaces). Suppose a finite sequence of exact operations $e_1, \dots, e_k$ is locally correct, all preconditions hold when used, and each produced target is an admissible source for the next operation. Starting at a specified x_0, the accepted final value is

$$x_k = f_{e_k}(f_{e_{k-1}}(\cdots f_{e_1}(x_0, c_1) \cdots, c_{k-1}), c_k).$$

If a subsequence is to be inverted, each inverse must have its stated retained data and inverse hypotheses. Forward composition alone implies neither injectivity nor reversibility.

Proof. For one operation the equality is its local correctness statement. Assume the equality through step j. Target membership and the next precondition make $f_{e_{j+1}}(x_j, c_{j+1})$ defined. Local correctness identifies the next accepted value with that mathematical value. Induction proves the composition. The final assertion follows because a many-to-one operation can occur in a perfectly correct composition; Proposition 2.2 supplies the criterion for recovery. □

The relevant operations and their proof obligations are as follows.

Operation	Required data and mathematical meaning
Marked geometry to measurement	Fixed construction, scale, center, frame, and selected lengths; Proposition 2.1
Measurement to normalized scalar	Positive scale and the specified symbolic length expressions; absolute scale and placement are not recovered from the ratio
Affine observation to integer register	Declared coordinate $d = v - 1$, integer lattice, floor rounding, and residual reset; Section 3
Rational map to chosen state	Explicit seed; the scalar constant does not determine a seed
State to ordered record	Quotient and remainder equations, adjacency, terminal zero, and first-zero convention; Theorem 3.2
Record to marked geometry	Positive resolution h, retained placement and construction; computed scale $a = hd$; Theorem 6.1
Map to iterated preimage	Base, iteration depth, and optional domain bound; Theorem 5.3
Certificate to angular observable	Actual node moment identities and specified matrix and radius; Theorems 8.1–8.3
Geometry to sampled motion or field	Numerical coordinates, chosen ambient axes, center, deformation parameters, and explicit instantaneous rates; Section 7

There are two distinct links between the real and rational maps. Defining F is a choice of arithmetic model. Theorems 4.2 and 6.1 prove its agreement with the real floor model on stated domains. Neither link asserts $q = \lambda$. A path containing numerical sampling makes a numerical evaluation claim at that step; relabeling its output cannot make it an exact geometric theorem.

The executable interface in the companion uses a closed, versioned envelope with fields `schema`, `type`, `domain`, `exactness`, and `value`. Rational numbers are reduced numerator-denominator pairs with positive denominator. Exact integer fields reject booleans and floating-point replacements. Each operation has a fixed context schema, and each submitted output is compared with a recomputed result. Unknown operations, extra fields, unsupported versions, and free-text substitutes for evidence are rejected. A saved path retains its initial value, operation names, contexts, and outputs, so the reader can repeat the checks at every step. These are executable specifications of the theorem's hypotheses, with the

implementation boundary recorded in Section 11.

10.2 The finite certificate and its interpretations

Fix the integer constants of Section 1, the marked template of Section 2, and the twelve-node rule of Theorem 8.2. Define the finite predicate $\mathcal{C}$ to require the following data and checks.

1. The integer and rational inequalities in Appendix A for the real parameter enclosure, the rational-separator determinant and ordering, and the boundary evaluations in Theorem 4.2.
2. Exactly one admissible first-zero record for each seed of D, arranged in the 32 specified disjoint blocks, with every seed bound, remainder, reverse divisibility, terminal condition, and depth at most ten checked.
3. The specified twelve-node inventory and weights, and the actual moment identities for every one of the 56 monomials through degree five.

The analytic interpretation of the rational enclosure uses the alternating-series and trigonometric argument in Appendix A. The interpretation of the monomial data uses the sphere measure and spanning proof in Theorem 8.1. Thus $\mathcal{C}$ is a finite arithmetic predicate whose consequences are supplied by proved mathematics.

Theorem 10.2 (Exact arithmetic certification of geometric reconstruction and transport). If $\mathcal{C}$ holds, the following statements hold with their indicated hypotheses.

(a) For every $h > 0$, admissible placement κ, and seed $d_0 \in D$, the retained ordered record recovers the entire marked registered construction and each registered intermediate state. Its orbit reaches zero in at most ten steps. The real and rational floor rules agree on that orbit, and for every $n \geq 0$ its distance from the continuously contracted construction is

$$d_H(\Gamma_\kappa(h\lambda^n d_0), \Gamma_\kappa(hd_n)) = hR_\Gamma(\lambda^n d_0 - d_n),$$

with the strict bound of Theorem 6.2 when $n \geq 1$.

(b) The certified rule integrates every real polynomial of degree at most five on S^2. In particular it gives the affine angular energy and velocity second moment for every real 3×3 matrix and every radius. For the symmetric traceless radially localized family it also gives the exact angular kinetic-energy density in the cutoff region, as stated in Theorem 8.6.

(c) For every finite dimension $m \geq 3$, smooth positive schedule satisfying Theorem 7.4, compact marked set, and viscosity $\nu > 0$, the associated volume-compensating deformation is realized by the smooth finite-energy forced solution of Theorem 7.6. In dimension three its similarity subfamily transports the entire marked outline; the matrix subfamily has the relative scalar observation in Proposition 7.2. The force is constructed with the prescribed motion and vanishes near the selected trajectories.

(d) Finite paths of these exact interfaces preserve their specified meanings whenever the local correctness, precondition, and retained-data hypotheses of Theorem 10.1 hold. Numerical paths retain their explicitly numerical status.

Proof. Condition 1, Lemma 4.1, and Appendix A prove Theorem 4.2 on the invariant domain. Condition 2 and Euclidean division reconstruct every stated orbit, and the fixed interval partition makes the conclusion exhaustive on D. Proposition 2.1 recovers the marked geometry for arbitrary retained h, κ. Theorem 6.2 then gives the exact distance and bound, proving (a). Condition 3 and the monomial spanning argument give degree-five exactness for every real coefficient choice. The matrix identities in Theorem 8.3 and the degree-four reduction in Theorem 8.6 give (b). The analytic construction

in Theorems 7.4 and 7.6 proves (c), while Propositions 7.1 and 7.2 identify its geometric and scalar observations. Theorem 10.1 proves (d). □

Part (c) is an analytic theorem about the explicit deformation family; it does not require enumerating fluid states and is logically independent of the finite moment certificate. Its conjunction with (a) and (b) places exact observations and reproducible calculations on a rigorously defined geometric realization. Similarly, part (d) specifies the composition obligation; it does not assert that arbitrary submitted software has met it. The forward observation maps need not be injective, and the constructed force need not coincide with a force prescribed by a separate physical model. These distinctions are part of the precise mathematical result.

11 Reproducible arithmetic and mathematical models

11.1 Exact certificate and artifact verification

The electronic companion [18] contains `omega_reference_checker.py`, the unchanged exact checker also distributed previously as `verify_paper8.py`. It uses Python's standard library and arbitrary-precision integers and rational numbers. It constructs the rational parameter enclosure, checks the first floor disagreement by both separator and floor-sum arguments, reconstructs the full finite seed domain, and computes the twelve-node moments in $\mathbb{Q}(\sqrt{5})$.

Field elements are pairs (a, b) representing $a + b\sqrt{5}$, where $a, b \in \mathbb{Q}$. Their multiplication and inverse are

$$(a, b)(c, d) = (ac + 5bd, ad + bc), \qquad (c, d)^{-1} = \frac{(c, -d)}{c^2 - 5d^2}.$$

For a nonzero pair the denominator is nonzero by irrationality of $\sqrt{5}$. Equality is pair equality. Odd-degree node sums vanish before normalization; the even-degree calculations remain in this field. No floating-point node evaluation controls moment acceptance.

The first-zero block bounds are inclusive:

$$I_j = \{1024j, \dots, 1024(j + 1) - 1\}, \qquad 0 \le j < 32.$$

For each $d \in D$, Euclidean division by 1024 gives its unique block index and row position. A block must contain exactly 1024 records, and the seed at row i must be $1024j + i$. Together with the row predicate of Appendix B, these conditions prove complete coverage without duplication. The program validates serialized and parsed records before publishing its result directory. Failed checks raise explicit errors; a successful artifact is not written after a failed law.

The separate reader `verify_external_bundle.py` validates the existing files in a submitted certificate directory. It checks exact file inventory, duplicate JSON keys, every stored row, coverage, block hashes, declared summary statistics, and recomputed arithmetic and moment claims. It reads the submitted ledger bytes rather than regenerating replacement ledgers before acceptance. Its routines are shared with the reference checker, so this is a separately executable reader, not an independently developed third-party implementation.

Executed verification	Result and scope
Exact seed certificate	32,768 rows in 32 disjoint blocks
First-zero transitions	300,871; greatest depth 10

Executed verification	Result and scope
Parameter boundary	First failure at seed 42,189,065; two exact arithmetic derivations
Fiber checks	Bases 0 through 50,000 and remote period endpoints
Cubature	56 monomials through degree five; exact degree-six failure witness
Reference negative controls	17 malformed cases rejected; normal and optimized executions agreed
Integrated interface and bundle tests	18 test methods passed normally and with optimization enabled
Model tests	Nine test methods passed; numerical dimensions 3, 4, 5, and 8 exercised
Saved interface examples	Eight paths exported; main and five-dimensional paths replayed from files
Existing bundle reader	All blocks accepted; an altered row was rejected even with an updated block hash
Symbolic field checks	The supplied 3D and 5D expressions simplified to zero divergence under SymPy 1.14.0

The reference negative controls cover invalid types, bounds, seed or terminal values, changed or reordered remainders, omissions, duplicates, wrong block indices, and zero padding. The integrated tests additionally exercise exactness and domain relabeling, omitted placement, inconsistent scales and frames, fabricated outputs, mutated saved paths, invalid JSON tokens, and numerical-output tampering. An integer as large as 10^{100} was exercised outside D for reconstruction; that example does not enlarge the real-floor agreement theorem. An exhausted explicit iteration budget raises failure rather than claiming termination. The general termination proof is strict integer descent.

The canonical certificate `sample_certificate/CERTIFICATE.json` has SHA-256

`32bcc2fefc44d0fe19575683b64fd67ddcd32327aeff32ab14cc988a507f4a79`

The reference checker has SHA-256

`dbf0650cf80c9e618684d98292bf7592baadadc51bd2eccf15b950392dcd5c8b`

The certificate records all 32 block hashes. These identify this serialization, rather than the earlier reported Grok fiber artifacts. A checksum binds a claim to particular bytes; validity still requires the mathematical predicates. The source archive includes a manifest for the complete release. The publication audit additionally reproduced all 32,768 rows and 56 moments using a separate audit script, rejected 662 mutated outputs across all 18 operations, and added three numerical boundary regression tests. Original and revised execution records are distinguished in the accompanying audit report.

The first forward product Qd exceeding a signed 64-bit integer occurs at $d = 27132$. Therefore fixed-width multiplication is insufficient even throughout D. Conversion to arbitrary precision must occur before multiplication. Mathematical reconstruction has no finite machine-capacity hypothesis; implementations must choose an adequate representation.

11.2 Implemented interface and modeling coverage

The version 1.1 interface provides 18 registered operations and 19 payload families. Its exact geometry records allow rational scales, rational centers, and exactly orthonormal rational frames in each supported finite dimension $m \geq 3$. This is an executable subset of the arbitrary-real placement theorem. The fixed

construction's transcendental constants are represented by a prescribed symbolic vocabulary. Arbitrary expression strings are not executed as mathematical evidence.

The numerical and symbolic modules implement the following objects.

Module or artifact	Content and interpretation
`omega_models.py`	Exact integer orbits, inverse records, fiber endpoints, layers, and rational sphere moments; numerical outline, deformation, cutoff field, pressure, pathlines, spiral action, and optional recursive sampling
`omega_symbolic_field.py`	Coordinate formulas for the localized velocity, divergence, pressure, Laplacian, and constructed forcing in a selected finite dimension
`omega_interface/`	Closed data envelopes, local preconditions, recomputed transitions, and saved-path verification
`sample_certificate/`	Complete exact seed ledgers and the canonical certificate
`replay_examples/`	Arithmetic, geometry, fiber, parameter, cubature, and numerical-model paths
`OMEGA_Model_Explorer.html`	Offline numerical exploration of the declared models

The modeled outline is a marked curve, not a solid body with an unspecified interior. The velocity and pressure formulas are defined on the full ambient space once the schedule and cutoff are supplied. Sample grids are evaluations of that field. The included higher-dimensional demonstration exports all coordinates but samples a three-coordinate slice of the ambient field; it is not a volumetric enumeration of $\mathbb{R}^m$.

Numerical deformation acts in the chosen first two ambient coordinates and about the retained center. An arbitrary marked frame must be deliberately aligned with the desired generator, or the whole field must be transformed by the rigid-placement formula of Section 7. Field sampling uses explicit instantaneous γ, β and cutoff radii. A single position parameter pair (c, η) does not determine these derivatives, and a few sampled points do not establish containment of a complete trajectory tube.

The model's smooth schedule uses a flat step from zero to one. It is constant before and after the specified interval and smooth through its endpoints, although it may begin changing immediately inside the interval. For literal constant neighborhoods as stated in Theorem 7.4, place that active interval strictly inside a larger motion interval. The theorem's support and smoothness hypotheses are then satisfied. The numerical pathline routine uses fourth-order Runge–Kutta and supplies no certified integration-error bound. The exact core trajectory is the relative map in Theorem 7.3 or 7.6.

The numerical implementation screens the deformation eigenvalue condition number: it rejects parameters when the binary64 machine epsilon times the spectral condition number exceeds 10^{-8}. This conservative guard prevents the demonstrated loss of the small eigenvalue under strong anisotropy; it is not a certified accuracy bound. Flat-step tails are evaluated at their binary64 underflow limits to avoid a spurious nonfinite derivative. Neither implementation restriction changes the analytic theorems.

Binary64 samples have the status `numerical_approximation`. Their replay is relative to the recorded numerical environment; it is not a proof of hardware-independent exact geometry. The observed environment was Python 3.12.14, NumPy 2.3.5, and SymPy 1.14.0. The explorer's interaction logic was checked, but a full browser visual inspection was not completed. No scientific claim in the paper depends on the appearance of its rendering.

11.3 Reproduction instructions

The source archive supplies the complete active code, dependencies, exact ledger files, interface examples, and the mathematical source of this paper. From its `companion` directory, the following commands reproduce the principal checks; newly generated output directories must not already exist.

```
python omega_reference_checker.py --out new_certificate
python -O omega_reference_checker.py --out optimized_certificate
python verify_external_bundle.py sample_certificate
python test_interface.py
python -O test_interface.py
python test_omega_models.py
python test_numerical_boundaries.py
python -m omega_interface --verify-path replay_examples/main.json
python run_integration.py --out new_paths
python omega_models.py --out model_3d --dimension 3
python omega_models.py --out model_5d --dimension 5
python omega_symbolic_field.py --dimension 3 --out field_3d.txt
```

The exact arithmetic and interface require Python 3.10 or later. Install the pinned numerical and symbolic dependencies from `requirements.txt` before running the full command list: the integrated interface tests and path exporter also exercise the NumPy adapters. The exact checker, existing-bundle reader, and exact interface operations alone use the standard library. Dimension is an explicit parameter; computational cost, numerical range, and memory constrain actual runs even though Theorem 7.6 holds for every finite $m \geq 3$.

No new Lean build is claimed. Earlier supplied Lean sketches included unproved analytic declarations, assumed parameter enclosures, and predicates that did not yet encode all the stated equations. Those sketches are not premises of the present paper. A formal version must define the actual objects and prove the displayed implications, without replacing them by an assumption or placeholder. The present result consists of the complete mathematical arguments, their specified finite predicates, and reproducible implementation evidence.

Appendix A Exact parameter enclosure and a second boundary proof

A.1 Rational enclosure of the real parameter

Let $R = \mathrm{isqrt}(5 \cdot 10^{160})$. The integer-square inequalities give

$$R/10^{80} < \sqrt{5} < (R+1)/10^{80}.$$

The inequalities are strict because 5 is not a square of a rational number. For $b = 5,239$, define the exact rational numbers

$$A_b = \sum_{j=0}^{63} \frac{(-1)^j}{(2j+1)b^{2j+1}}, \qquad B_b = A_b + \frac{1}{129b^{129}}.$$

The alternating series with decreasing terms gives

$$A_b < \arctan(1/b) < B_b.$$

For completeness, Machin's identity on the required branch follows without numerical approximation. If $\theta = \arctan(1/5)$ and $\eta = \arctan(1/239)$, then

$$\tan(2\theta) = 5/12, \quad \tan(4\theta) = 120/119, \quad \tan(4\theta - \eta) = 1.$$

For $x > 0$, integrating $1/(1+t^2)$ on $[0, x]$ gives $x/(1+x^2) < \arctan x < x$. These bounds imply $0 < 4\theta - \eta < 4/5 < \pi/2$. Thus $4\theta - \eta = \pi/4$ and

$$16A_5 - 4B_{239} < \pi < 16B_5 - 4A_{239}.$$

Write φ_-, φ_+ for the bounds obtained from the square-root interval and π_-, π_+ for the last interval. The function $x/(x+y)$ increases in $x > 0$ and decreases in $y > 0$. Therefore

$$\frac{\varphi_-}{\varphi_- + \pi_+} < \lambda < \frac{\varphi_+}{\varphi_+ + \pi_-}.$$

Exact cross multiplication yields

$$\frac{Q}{S} < \frac{3399497713447789295767028093 21}{10^{30}} < \lambda < \frac{33994977134477892957670280932 2}{10^{30}}.$$

The companion constructs every rational bound above and verifies the cross multiplications. Thus the enclosure is a reproducible rational inequality, not a floating-point estimate or an assumed decimal expansion.

A.2 A terminating floor-sum calculation

Define

$$\mathcal{H}(n, m, a, b) = \sum_{i=0}^{n-1} \left\lfloor \frac{ai + b}{m} \right\rfloor$$

for nonnegative integers n, a, b and positive m. Write $a = um + a_0$ and $b = vm + b_0$ with $0 \le a_0, b_0 < m$. Then

$$\mathcal{H}(n, m, a, b) = u\frac{n(n-1)}{2} + vn + \mathcal{H}(n, m, a_0, b_0).$$

For the remaining sum put $k, r = \mathrm{divmod}(a_0 n + b_0, m)$. If $k = 0$, the sum is zero. Otherwise $a_0 > 0$ and

$$\mathcal{H}(n, m, a_0, b_0) = \mathcal{H}(k, a_0, m, r).$$

To prove the latter identity, count integer horizontal levels under the graph:

$$\mathcal{H}(n, m, a_0, b_0) = \sum_{j=1}^{k} \left(n - \left\lceil \frac{mj - b_0}{a_0} \right\rceil \right).$$

Since $a_0 n + b_0 = mk + r$, the summand equals $\lfloor (m(k-j) + r)/a_0 \rfloor$. Summing those terms gives $\mathcal{H}(k, a_0, m, r)$. A possible empty top level contributes zero. The new modulus a_0 is smaller than m, so the recurrence terminates by the Euclidean algorithm. It uses only integers.

With $n = d_*$, $m = 10^{30}$, and L as above, the recurrence gives

$$\mathcal{H}(d_*, 10^{30}, L+1, 0) = \mathcal{H}(d_*, S, Q, 0) = 302541195258184.$$

For every nonnegative index, the first sum's summand is at least the second's, because $(L+1)/10^{30} > q$. Their equal totals therefore imply equality at every index $0 \le d < d_*$. Squeezing λ between q and the upper bound proves the floor agreement on that entire interval. At d_*, direct division gives

$$\left\lfloor \frac{Ld_*}{10^{30}} \right\rfloor = \left\lfloor \frac{(L+1)d_*}{10^{30}} \right\rfloor = 14342163,$$
$$\lfloor Qd_*/S \rfloor = 14342162, \qquad Qd_* \bmod S = 999999976452570.$$

This is a second exact proof of the first boundary, independent of the rational-separator argument once the same real-parameter enclosure has been established. It does not enumerate 42 million separate ledgers.

Appendix B Checker soundness and reproducibility boundary

The essential row checker can be stated without a programming language. Given a claimed seed d, terminal zero, and a finite ordered list $(\rho_0, \dots, \rho_{k-1})$, it performs the following operations.

1. Require integer types, $0 \le \rho_j < S$, and the specified length bound.
2. Starting from $x_k = 0$, form $v_j = Sx_{j+1} + \rho_j$ in reverse order.
3. Require $Q \mid v_j$ and put $x_j = v_j/Q$. Require $x_j > x_{j+1}$.
4. Require $x_0 = d$ and, for an empty list, require $d = 0$.

Proposition B.1 (Row soundness). Acceptance implies that the claimed seed generates exactly the reconstructed first-zero orbit under F, with exactly the supplied ordered remainders.

Proof. At every reverse step, $Qx_j = Sx_{j+1} + \rho_j$ and $0 \le \rho_j < S$ imply $F(x_j) = x_{j+1}$ by Euclidean division. Starting at terminal zero and requiring strict increase in reverse gives nonnegative states and excludes any earlier zero in forward order. The equality $x_0 = d$ binds the record to its claimed seed. Thus all forward transitions and their remainders are precisely those claimed. The empty record represents only the already terminal seed zero. □

For the block checker, require exactly 1024 records at positions $i = 0, \dots, 1023$, with claimed seed $1024j + i$, and invoke this row checker. Require each of the 32 predetermined block indices exactly once. The integer partition proof in Section 11 then makes acceptance exhaustive on D.

The finite predicate also needs the analytic parameter to be bound to the intended rational map. Appendix A supplies that binding. For cubature, the predicate fixes the node construction and weights, checks the full required exponent set, and compares the actual calculated moments with the stated exact sphere moments. Merely checking a list of names, an empty table, or numbers regenerated from the expected answers would not establish those conditions.

A certificate hash is useful for identifying a particular replay target, but is not a replacement for any of these checks. Similarly, regression on 50,001 or 100,000 bases corroborates an implementation of the fiber law; its validity for all bases follows from the proof of Theorem 5.1, including the equality endpoints.

Appendix C Source and version record

The OMEGA references identify the source corpus supplied for this project: the quantized contraction and ordered-remainder papers, the frozen resolution specification, the radial integration and cubature papers, the scalar reference report, and the marked-geometry manuscript. Their supplied identifiers and working labels are retained in the bibliography. A working revision number and a repository deposit version identify different things and should not be interchanged.

This paper defines and proves its used statements explicitly. The website is a contextual index, not a premise of any proof. Present availability of every website route, reserved identifier, or earlier deposit is not represented as independently verified. In particular, the supplied 4096-cell integral certificate and unrelated higher-degree searches have not been recertified by the twelve-node calculation here.

The current electronic companion retains the corrected `OMEGA_INTERFACE/1.1` schema and unchanged exact reference certificate. The publication audit updates the numerical modeling module to 0.1.1 and adds numerical boundary regressions; `PUBLICATION_AUDIT_CHANGES.md` records those changes. Its source, fixed constants, predicates, and execution record identify the implemented claims. The archived `original_1_0` directory is retained only as superseded provenance; acceptance and replay use the top-level version 1.1 implementation. Historical scaffold and contract files document preparation of that release; the present manuscript supplies the final mathematical exposition.

The supplied fluid and spiral manuscripts are sources for the constructions in Sections 7 and 9. The higher-dimensional localization, exact homothety distance, and localized angular-observable statements are proved here in the forms used. The general methods of Euclidean division, rational separators, spherical designs, compact-set contraction, and proof by finite certificates are established mathematics and are credited accordingly.

Appendix D Minimal exact reconstruction algorithm

The following standard-library code states the central first-zero record algorithm in a compact form. The complete companion adds serialization, block coverage, parameter enclosures, node arithmetic, path checking, and models. Integer bounds and types are checked explicitly; the mathematical proof of acceptance is Proposition B.1.

```
Q = 339949771344778
S = 10**15

def require(condition, message):
    if not condition:
        raise ValueError(message)

def first_zero_record(seed):
    require(type(seed) is int and seed >= 0, "invalid seed")
    d = seed
    remainders = []
    while d:
        e, rho = divmod(Q * d, S)
        remainders.append(rho)
        d = e
    return {"seed": seed, "terminal": 0,
```

```
            "remainders": remainders}

def verify_record(record, max_depth=None):
    require(type(record) is dict, "record must be a dict")
    require(set(record) == {"seed", "terminal", "remainders"},
            "wrong fields")
    seed, terminal = record["seed"], record["terminal"]
    rs = record["remainders"]
    require(type(seed) is int and seed >= 0, "invalid seed")
    require(type(terminal) is int and terminal == 0,
            "terminal must be zero")
    require(type(rs) is list, "remainders must be a list")
    if max_depth is not None:
        require(type(max_depth) is int and max_depth >= 0,
                "invalid depth bound")
        require(len(rs) <= max_depth, "depth exceeded")
    d = 0
    reverse_orbit = [d]
    for rho in reversed(rs):
        require(type(rho) is int and 0 <= rho < S,
                "invalid remainder")
        predecessor, residue = divmod(S * d + rho, Q)
        require(residue == 0, "nonintegral reverse step")
        require(predecessor > d, "not a first-zero step")
        d = predecessor
        reverse_orbit.append(d)
    require(d == seed, "seed mismatch")
    return list(reversed(reverse_orbit))

def iterated_fiber(base, depth):
    require(type(base) is int and base >= 0, "invalid base")
    require(type(depth) is int and depth >= 0, "invalid depth")
    lo, stop = base, base + 1
    for _ in range(depth):
        lo = (S * lo + Q - 1) // Q
        stop = (S * stop + Q - 1) // Q
    return lo, stop - 1
```

The generation loop terminates by strict descent of nonnegative integers. The verification loop terminates because the submitted list is finite. On D, the optional depth bound is ten; outside D, a different bound must be established or omitted. The fiber routine takes a number of exact arithmetic steps linear in the requested depth, with integer bit cost depending on endpoint size. It represents potentially large preimage sets by exact endpoints rather than listing every predecessor.

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Preparation disclosure

This manuscript was prepared with ChatGPT assistance for mathematical exposition, proof development, computational checks, and document production. The author is responsible for its use and publication. The stated execution record is reproducibility evidence at its specified scope; preparation does not constitute external peer review or proof-assistant verification.