

Proof Dossier: Positive Icosahedral Cubature on Fixed Class-I Grids

Finite exact extension through frequency 21

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1 Setup

Let $\phi = (1 + \sqrt{5})/2$. Let $X_f \subset \mathbb{S}^2$ be the complete radial Class-I icosahedral grid of frequency f , obtained by barycentric subdivision of every icosahedral face followed by radial projection and global deduplication. Then $|X_f| = 10f^2 + 2$.

Let H_3 denote the full icosahedral reflection group. On a fixed face choose vertices

$$v_1 = (0, 1, \phi), \quad v_2 = (-1, \phi, 0), \quad v_3 = (1, \phi, 0).$$

Every H_3 -orbit in X_f has a unique representative

$$x_{a,b,c} = \frac{av_1 + bv_2 + cv_3}{\|av_1 + bv_2 + cv_3\|}, \quad a \geq b \geq c \geq 0, \quad a + b + c = f.$$

Thus the orbit set is indexed by partitions of f into at most three parts.

The invariant ring of H_3 is polynomial on homogeneous generators of degrees 2, 6, 10. On $\mathbb{S}^2$ the degree-two generator is constant, so invariant polynomial restrictions are generated by a degree-six invariant U_6 and a degree-ten invariant U_{10} . We take U_6 to be the normalized harmonic I_6 used in the underlying Class-I analysis, and define

$$U_{10}(x) = \sum_{v \in V} P_{10}\left(\frac{x \cdot v}{R}\right), \quad P_{10}(t) = \frac{46189t^{10} - 109395t^8 + 90090t^6 - 30030t^4 + 3465t^2 - 63}{256},$$

where V is the set of twelve icosahedral vertices and $R^2 = \phi + 2$.

Lemma 1 (Invariant reduction). *Suppose orbit masses W_λ are constant on H_3 -orbits and sum to one. The resulting rule is exact for every spherical polynomial of degree at most t if and only if*

$$\sum_{\lambda} W_{\lambda} U_6(x_{\lambda})^r U_{10}(x_{\lambda})^s = \int_{\mathbb{S}^2} U_6^r U_{10}^s d\sigma$$

for every $r, s \geq 0$ satisfying $6r + 10s \leq t$.

Proof. Sobolev's invariant-cubature theorem reduces exactness of an H_3 -invariant rule to exactness on H_3 -invariant polynomials. Since $\mathbb{R}[x, y, z]^{H_3}$ is generated in degrees 2, 6, 10, restriction to the unit sphere replaces the degree-two generator by 1, leaving precisely the displayed monomials. $\square$

Lemma 2 (Exact quadratic-field closure). *For every orbit representative $x_{a,b,c}$,*

$$U_6(x_{a,b,c}), U_{10}(x_{a,b,c}) \in \mathbb{Q}(\sqrt{5}).$$

Moreover every required spherical moment belongs to $\mathbb{Q}(\sqrt{5})$.

Proof. The unprojected representative has coordinates in $\mathbb{Q}(\sqrt{5})^3$. Since U_6 and U_{10} are homogeneous of even degrees, radial normalization introduces only powers of the squared norm, which lies in $\mathbb{Q}(\sqrt{5})$. For a monomial, normalized spherical area satisfies

$$\int_{\mathbb{S}^2} x^{2i} y^{2j} z^{2k} d\sigma = \frac{(2i-1)!!(2j-1)!!(2k-1)!!}{(2i+2j+2k+1)!!},$$

and any monomial with an odd exponent has mean zero. Expansion therefore keeps all moment arithmetic in $\mathbb{Q}(\sqrt{5})$. $\square$

2 The frequency-two rule

Let V be the twelve vertices and E the thirty radially projected edge midpoints. Define

$$Q_2[p] = \frac{5}{252} \sum_{x \in V} p(x) + \frac{8}{315} \sum_{x \in E} p(x).$$

Theorem 1. Q_2 has algebraic degree exactly 9.

Proof. The orbit masses are

$$12 \frac{5}{252} = \frac{5}{21}, \quad 30 \frac{8}{315} = \frac{16}{21},$$

and sum to one. For the degree-six invariant,

$$U_6(V) = \frac{132}{25}, \quad U_6(E) = -\frac{33}{20},$$

hence

$$\frac{5}{21} \frac{132}{25} + \frac{16}{21} \left(-\frac{33}{20} \right) = 0.$$

There is no other nonconstant invariant condition below degree ten, so the invariant reduction proves exactness through degree nine.

For exact failure at degree ten, direct substitution into the zonal sum gives

$$U_{10}(V) = \frac{988}{625}, \quad U_{10}(E) = -\frac{247}{160}.$$

Since U_{10} is a nonconstant spherical harmonic, its spherical mean is zero, whereas

$$Q_2[U_{10}] = \frac{5}{21} \frac{988}{625} + \frac{16}{21} \left(-\frac{247}{160} \right) = -\frac{4199}{5250} \neq 0.$$

Thus the degree is exactly nine. $\square$

For unnormalized surface area, multiply every probability weight by 4π : the per-node weights become $5\pi/63$ on V and $32\pi/315$ on E .

3 Exact finite-frequency extension

Define

$$\tau_f = \begin{cases} 3f+2, & f \text{ odd}, \\ 3f+3, & f \text{ even}. \end{cases}$$

Theorem 2 (Exact certified block). *For every integer $2 \leq f \leq 21$ there exist strictly positive weights on every node of X_f , constant on H_3 -orbits, that integrate every spherical polynomial of degree at most τ_f exactly.*

Proof. For each f , form the exact moment matrix

$$A_{(r,s),\lambda} = U_6(x_\lambda)^r U_{10}(x_\lambda)^s, \quad 6r + 10s \leq \tau_f,$$

and the exact spherical-moment vector

$$b_{r,s} = \int_{\mathbb{S}^2} U_6^r U_{10}^s d\sigma.$$

All entries are represented as pairs $(a, b) \in \mathbb{Q}^2$ meaning $a + b\sqrt{5}$. The attached certificate supplies an exact solution of $AW = b$ for every $f = 2, \dots, 21$. The verifier checks, using rational arithmetic only, that (i) every moment identity holds exactly, (ii) $\sum_\lambda W_\lambda = 1$ exactly, and (iii) every $W_\lambda > 0$ by exact sign comparison in $\mathbb{Q}(\sqrt{5})$. By the invariant-reduction lemma, these identities prove cubature exactness through τ_f . $\square$

f	$ X_f $	orbits	moment conditions	τ_f	minimum orbit mass
2	42	2	2	9	0.2380952381
3	92	3	3	11	0.0809836277
4	162	4	4	15	0.0538689578
5	252	5	5	17	0.0305198337
6	362	7	7	21	0.0217485293
7	492	8	8	23	0.0153277755
8	642	10	10	27	0.0120867002
9	812	12	11	29	0.0185370981
10	1002	14	14	33	0.0076756094
11	1212	16	15	35	0.0103682232
12	1442	19	18	39	0.0063606773
13	1692	21	20	41	0.0066051071
14	1962	24	23	45	0.0054550724
15	2252	27	25	47	0.0084821346
16	2562	30	29	51	0.0065375477
17	2892	33	31	53	0.0065448870
18	3242	37	35	57	0.0057291345
19	3612	40	37	59	0.0054331943
20	4002	44	42	63	0.0046067688
21	4412	48	44	65	0.0045726956

The machine verification reports 383 exact invariant moment equations across the twenty frequencies $f = 2, \dots, 21$, all passing.

4 Explicit low-frequency certificates

For $f = 3$, with orbit order $(3, 0, 0), (2, 1, 0), (1, 1, 1)$, one exact positive solution is

$$\begin{aligned} W_{300} &= \frac{26375}{201432} - \frac{375}{16786}\sqrt{5}, \\ W_{210} &= \frac{1570589}{1007160} - \frac{4731}{11990}\sqrt{5}, \\ W_{111} &= -\frac{28971}{41965} + \frac{17496}{41965}\sqrt{5}. \end{aligned}$$

These sum exactly to one and are approximately $(0.0809836277, 0.6771184732, 0.2418978991)$.

For $f = 9$ the system through degree 29 has one free orbit mass. Taking $W_{333} = 1/30$ gives an exact strictly positive solution in $\mathbb{Q}(\sqrt{5})$; the full vector is recorded in the certificate file.

5 All-frequency target

Conjecture 1. *For every $f \geq 2$, the fixed Class-I grid X_f admits strictly positive, H_3 -orbit-constant weights of degree at least τ_f .*

Equivalently, if

$$v_\lambda = (U_6(x_\lambda)^r U_{10}(x_\lambda)^s)_{6r+10s \leq \tau_f}, \quad b = \left(\int_{\mathbb{S}^2} U_6^r U_{10}^s d\sigma \right)_{6r+10s \leq \tau_f},$$

then strict positivity is equivalent to

$$b \in \text{relint conv}\{v_\lambda\}.$$

This is the appropriate strict-positivity version of the Farkas-dual formulation. Nonnegative weights correspond to membership in the convex hull; strictly positive weights correspond to relative-interior membership.

No all-frequency proof is claimed here. Likewise, maximality of τ_f is not claimed as a theorem in this dossier.

6 Literature boundary

The symmetry reduction is classical: Sobolev’s criterion is discussed by Nozaki–Sawa. Icosahedrally invariant quadrature construction is classical, including McLaren, Konyaev, Ahrens–Beylkin, and Popov. Gräf–Kunis–Potts study nonnegative weights on general scattered spherical nodes. The finite certificate theorem above is therefore stated only as an exact extension for the specific fixed radially projected Class-I grids; no priority claim is made without a dedicated historical search.

References

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