

OMEGA NAUGHT · CLASS-I CUBATURE

A New Exact Boundary at Frequency 22

Teaching and Verification Addendum for Omega Naught Class-I
Icosahedral Cubature

Exact result. The fixed radial Class-I grid X_{22} , containing 4,842 nodes, supports strictly positive weights of algebraic degree 69. No nonnegative choice of weights on those same nodes can reach degree 70.

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Exact arithmetic in $\mathbb{Q}(\sqrt{5})$. Python 3.12.13 and Julia 1.11.6 both CERTIFICATE PASS.

Canonical solution digests unchanged from the 24 August 2026 calculation.

Correction notice (v1.1)

This version supersedes the 24 August 2026 addendum PDF (SHA-256 72332ba4f97425b132fe8073aeccfd5450e19c8f62a3a654969401213dced7a1, 356694 bytes). That document selected the minimum positive degree-69 orbit after binary64 conversion and incorrectly named (22,0,0) with decimal 0.003874805888244737. Exact comparison in $\mathbb{Q}(\sqrt{5})$ gives (11,11,0). The unique negative forced mass remains on (21,1,0); only its printed decimal is corrected. Reproduction is now recorded for both Python and Julia. The theorem statement and both canonical solution digests are unchanged.

Corrected summary (decimals are orientation only)

Minimum positive orbit: (11,11,0)

Minimum orientation mass: 0.003874805888537519508839054528148877...

Unique negative orbit: (21,1,0)

Negative orientation mass: -0.006609660200017031631251308016577647...

Python 3.12.13: CERTIFICATE PASS

Julia 1.11.6: CERTIFICATE PASS

Python source SHA-256: ae615d262ecfc564aed119e5002c4fa9

c4fadf35812002426298caf7c6ab89bf

Julia source SHA-256: 5c45650b3baec9889204ac423f8ec62c

9cc0782a03e5c30a8467b9cb01a1efea

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1. What was learned

Paper VI v1.3 proves the original Class-I results for $1 \leq f \leq 9$. A post-publication computation is reported for $10 \leq f \leq 21$ without a publicly fixed certificate package on this site. The present calculation moves the exact boundary one frequency: $f = 22$ attains algebraic degree 69 with strict positivity, and 69 is the maximum nonnegative degree on the frozen node set X_{22} .

The target degree used for the sequence is

$$t_f = 3f+2 \text{ if } f \text{ is odd, and } t_f = 3f+3 \text{ if } f \text{ is even.}$$

At $f = 22$ one has $t_{22} = 69$. This is a statement about existence of strictly positive H_3 -orbit-constant rules of that degree, not a statement that the integer t_f itself is “positive.”

Theorem 1 (Frequency-22 exact boundary)

Let $X_{22} \subset S^2$ be the complete radially projected Class-I icosahedral grid of frequency 22, and normalize spherical area so that $\sigma(S^2) = 1$.

1. There are strictly positive weights on all nodes of X_{22} , constant on full icosahedral H_3 -orbits, whose cubature rule is exact for every spherical polynomial of degree at most 69.
2. No nonnegative weights on X_{22} can give a cubature rule exact for every spherical polynomial of degree at most 70.

Consequently the maximum algebraic degree attainable by nonnegative weights on the fixed node set X_{22} is exactly 69.

This is an exact computational theorem relative to the frozen definitions below. A web literature check found higher-degree icosahedrally invariant formulas whose nodes are allowed to move, so this addendum makes no blanket claim of being the first degree-69 spherical formula. The distinctive claim is about the fixed radial Class-I grid X_{22} , nonnegative weights, and exact arithmetic.

2. Start from ordinary weighted averages

A cubature rule on the sphere is a weighted average of function values, $Q[p] = \sum_{x \in X} w_x p(x)$. With normalized area, the true spherical average is $\int_{S^2} p(x) d\sigma(x)$, and $\int_{S^2} 1 d\sigma = 1$. The rule is exact through degree t if these two expressions agree for every polynomial $p(x,y,z)$ of total degree at most t , restricted to the unit sphere. The constant polynomial $p = 1$ forces $\sum w_x = 1$. Nonnegative weights keep the computed average between the smallest and largest sampled values. Strictly positive weights also ensure that no grid node is silently discarded.

Novice checkpoint

Degree 69 does not mean that one special polynomial was integrated. It means every polynomial built from x, y, z with total degree at most 69 is integrated exactly on the sphere. The theorem is not saying general smooth functions have zero error. It says their polynomial content through degree 69 has zero error.

3. The fixed Class-I grid

Let $\phi = (1+\sqrt{5})/2$ and choose one face of a regular icosahedron with vertices of the usual golden-rectangle construction, including $v = (-1, \phi, 0)$ and its H_3 images. For a positive integer f , subdivide each face into a triangular barycentric lattice of frequency f . A lattice point has nonnegative integer coordinates a, b, c with $a+b+c = f$. Radially project $p_{a,b,c} = a v_1 + b v_2 + c v_3$ to the unit sphere, repeat this on all faces, and deduplicate shared edges and vertices. The resulting node set X_f satisfies $|X_f| = 10f^2 + 2$. At $f = 22$ one has $|X_{22}| = 4,842$.

The full icosahedral reflection group H_3 has order 120. Symmetry moves most of the 4,842 nodes into large families called orbits. Every orbit has one representative with $a \geq b \geq c \geq 0$ and

$a+b+c = f$. Thus the orbits are indexed by partitions of f into at most three parts. For $f = 22$ there are exactly 52 such partitions.

Instead of one unknown per node, assign one orbit mass W_λ to each orbit λ . Every node in that orbit receives per-node weight $W_\lambda/|\lambda|$. The normalization is simply $\sum_{\lambda=1}^{52} W_\lambda = 1$. An orbit mass is the total weight carried by a whole symmetric family. If an orbit has 60 nodes and mass 0.12, each node has weight $0.12/60 = 0.002$. Confusing these two quantities changes the rule.

4. Why exact arithmetic stays possible

The coordinates contain $\sqrt{5}$, and radial normalization seems at first to introduce square roots of complicated norms. The moment equations use homogeneous invariants of even degree. Their normalization denominators are powers of the squared norm, so all values remain in $\mathbb{Q}(\sqrt{5}) = \{ a + b\sqrt{5} : a, b \in \mathbb{Q} \}$. The verifier stores $a + b\sqrt{5}$ as the rational pair (a, b) . Addition and multiplication are exact. For sign tests it never approximates $\sqrt{5}$: if a and b have opposite signs, it compares a^2 with $5b^2$, taking the signs of a, b into account.

Spherical monomial means are rational. For nonnegative even exponents the mean is a ratio of double factorials; any monomial with an odd exponent has mean zero. Expanding the invariant polynomials therefore keeps every required moment inside $\mathbb{Q}(\sqrt{5})$.

Exact claim types

- The equation $AW = b$ means equality of rational pairs, not a residual below a tolerance.
- The inequality $W_\lambda > 0$ means an algebraic sign proof in the chosen real embedding $\sqrt{5} > 0$, not a positive printed decimal.
- Printed decimals are explanatory metadata only.

5. The symmetry reduction

Define two invariant functions using the twelve icosahedral vertices V . If $R^2 = \varphi + 2$, set

$$U_6(x) = \sum_{v \in V} (x \cdot v)^6 / R^6, \quad U_{10}(x) = \sum_{v \in V} (x \cdot v)^{10} / R^{10}.$$

The invariant ring of H_3 is generated in degrees 2, 6, 10. On the unit sphere the degree-two generator is $x^2+y^2+z^2 = 1$, so the restricted invariant polynomials are spanned by $U_6^r U_{10}^s$ for $r, s \in \mathbb{Z}_{\geq 0}$. Sobolev's invariant-cubature principle says that an H_3 -invariant rule is exact through degree t if and only if it is exact on the invariant polynomials through degree t . Therefore one needs only the rows with $6r + 10s \leq t$.

For orbit representatives x_λ , the exact linear system is $A_{(r,s),\lambda} = U_6(x_\lambda)^r U_{10}(x_\lambda)^s$ and $b_{r,s} = \int_{S^2} U_6^r U_{10}^s d\sigma$, so $AW = b$. There are enormously many ordinary monomials of degree at most 69. Symmetry does not pretend they disappear. It proves that if the few invariant tests pass, averaging any error over the group forces every other test to pass as well.

6. The degree-69 construction

The number of invariant conditions is $N(t) = \#\{ (r,s) \in \mathbb{Z}_{\geq 0} : 6r+10s \leq t \}$. At $t = 69$, $N(69) = 49$. The exact moment matrix has 49 rows, 52 columns, and exact rank 49. Its solution set is therefore a three-dimensional affine space.

In the verifier's deterministic orbit ordering, the three free columns are (9,7,6), (8,8,6), and (8,7,7). Choose their orbit masses to be the rational numbers

$$\begin{aligned} W_{9,7,6} &= 11607197/394254842, \\ W_{8,8,6} &= 2312954/161188803, \\ W_{8,7,7} &= 7156114/472522843. \end{aligned}$$

Exact back-substitution determines the other 49 masses in $\mathbb{Q}(\sqrt{5})$. The verifier then proves $AW = b$, $\sum W_\lambda = 1$, and $W_\lambda > 0$ for every λ . Exact comparison of orientation masses, using the sign of differences in $\mathbb{Q}(\sqrt{5})$ rather than binary64 conversion, shows that the smallest orbit mass is at (11,11,0):

$$W_{11,11,0} \approx 0.003874805888537519508839054528148877... > 0.$$

The decimal is not the certificate; the exact $\mathbb{Q}(\sqrt{5})$ sign test is. The earlier addendum incorrectly named (22,0,0) after a binary64 comparison that reversed two nearby positive masses. That reporting error does not change any exact solution vector.

Quantity	Exact reported value
Nodes in X_{22}	4,842
H_3 -orbits	52
Invariant rows through degree 69	49
Exact rank	49
Affine freedom	3
Minimum positive orbit	(11,11,0)
Minimum orientation mass	0.003874805888537519508839054528148877...
Positive solution digest	0865572f76c7b8c9816ce937a79361c8 926abe785ac3f41b06e4f289665bf340

Table 1. Degree-69 construction data. Digests hash the canonical exact solution payload, not the printed decimals.

7. Why degree 70 fails

At degree 70, exactly three new invariant monomials appear: $U_6^{10} U_{10}$, $U_6^5 U_{10}^4$, and U_{10}^7 . Their exponent pairs are $(r,s) = (10,1)$, $(5,4)$, $(0,7)$, and each has total degree $6r+10s = 70$. Adding these rows changes the system from 49×52 to 52×52 . Exact elimination proves the matrix is nonsingular. Thus there is one and only one orbit-mass vector that could possibly reach degree 70. That vector has exactly one negative component:

$$W_{21,1,0} \approx -0.006609660200017031631251308016577647... < 0.$$

The exact object is an element of $\mathbb{Q}(\sqrt{5})$; its expression is very large, so the standalone verifiers recompute it rather than asking a reader to trust a rounded or typeset coefficient. No new invariant condition occurs at degree 71. All H_3 invariants have even degree, and the group contains central inversion. Therefore the degree-70 and degree-71 invariant systems are identical.

A dual proof that exposes the contradiction

Let A_{71} be the 52×52 matrix and let $e_\square$ select the orbit $\square = (21,1,0)$. Solve exactly $A_{71}^T c = e_\square$. The coefficients c define an invariant polynomial P of degree at most 70 whose sampled orbit

values are $P(x_\square) = 1$ and $P(x_\lambda) = 0$ for $\lambda \neq \square$. Its exact spherical mean is $\int_{S^2} P \, d\sigma = c^T b = W_\square < 0$. If nonnegative orbit masses integrated degree 70, they would instead give $Q[P] = \sum W_\lambda P(x_\lambda) = W_\square \geq 0$, contradicting the negative true mean. This is a Farkas-style separating certificate: one polynomial proves impossibility.

Novice checkpoint

The negative candidate weight is already enough once uniqueness is proved. The dual polynomial is stronger as an audit object: it turns the obstruction into a single inequality that every nonnegative rule would violate.

8. Why arbitrary weights do not escape the obstruction

The computation used orbit-constant weights, but the nonexistence conclusion applies to all nonnegative weights on the fixed grid.

Lemma 2 (Reynolds averaging / symmetrization)

If arbitrary nonnegative node weights on an H_3 -invariant node set are exact through degree t , averaging those weights over H_3 produces nonnegative orbit-constant weights that remain exact through degree t .

Proof. For each group element g , permute the node weights by g , then average the resulting 120 rules. Nonnegativity and total mass are preserved. The average is constant on orbits. Spherical measure is invariant under H_3 , so every permuted rule, and hence their average, integrates the same polynomial space exactly.

If any nonnegative degree-70 rule existed on X_{22} , the lemma would produce a nonnegative orbit-constant one. The dual certificate rules that out. This proves the second part of Theorem 1 without assuming the original weights respected symmetry.

9. A counting law that predicts the frontier

There is a useful asymptotic reason that the boundary appears near degree $3f$. The number of partitions of f into at most three parts satisfies $\#\text{orbits}(X_f) \sim f^2/12$, while the number of invariant moment rows satisfies $N(t) \sim t^2/120$. Equating the leading terms gives $t \approx \sqrt{10} f \approx 3.1623 f$. This is a dimension-count heuristic, not a proof of positivity or impossibility. It does explain why a target near $3f$ is plausible and why the linear systems become tight not far above it. At $f = 22$, $\sqrt{10} \cdot 22 \approx 69.57$. The positive degree-69 system has three degrees of freedom; the three new degree-70 rows make it square and force the negative mass.

Interpretation

Frequency 22 sits almost exactly at the invariant dimension threshold. The obstruction is not random numerical instability. It appears precisely when the last affine freedom is consumed by the next three invariant moments.

10. What this does and does not advance

Established now

- A strictly positive H_3 -orbit-constant rule of degree $t_{22} = 69$ exists on the fixed grid X_{22} .
- For $f = 22$, target degree 69 is not merely a lower bound; it is the exact maximum under nonnegative weighting of the fixed node set.
- The obstruction has an exact primal witness (one forced negative orbit mass at $(21,1,0)$) and an exact dual witness (a cardinal invariant polynomial with negative spherical mean).
- The result is reproducible with standard-library Python and Julia and exact rational-pair arithmetic. Both independent implementations print CERTIFICATE PASS and the same canonical solution digests.

Not established

- No all-frequency proof of existence of strictly positive H_3 -orbit-constant rules of target degree t_f for every $f \geq 23$ is supplied.
- Public certificate packages for $10 \leq f \leq 21$ are not yet released; those frequencies remain reported post-publication computations.
- No statement is made about cubature rules whose nodes may move away from the fixed Class-I grid.
- No claim is made that degree 69 is a record among all spherical cubature formulas; it is a sharp fixed-grid result.
- Machine verification is not a substitute for independent mathematical review, source publication, or historical priority research.
- External archival fixation (DOI / independent timestamp) is not claimed for this corrected package.

Research conjecture 3 (Positive fixed-Class-I sequence)

For every integer $f \geq 2$, the radial Class-I grid X_f admits strictly positive weights of degree at least t_f , where $t_f = 3f+2$ if f is odd and $t_f = 3f+3$ if f is even. The exact evidence now covers $f = 22$ as a separate additive result. The next direct test is $f = 23$, but the deeper goal is to replace one-frequency elimination with a uniform convex-geometric or asymptotic proof.

11. How to verify the theorem

Place either verifier in any directory with the stated runtime, then run one of:

```
python3 Omega_Naught_ClassI_f22_exact_verifier.py
julia OmegaNaughtClassIF22ExactVerifier.jl
```

Each program uses only the language standard library (Python: fractions, decimal, hashlib; Julia: SHA, Printf) and performs the following checks:

1. constructs the degree-6 and degree-10 invariants exactly;
2. computes their exact spherical moments;
3. enumerates all 52 orbit representatives of X_{22} ;
4. row-reduces the degree-69 system in $\mathbb{Q}(\sqrt{5})$;
5. substitutes the three stated rational free masses;
6. verifies all 49 identities, total mass 1, and 52 positive signs, selecting the minimum by exact sign of differences;

7. adds the three degree-70 rows and solves the forced square system;
8. verifies the single negative mass at (21,1,0);
9. builds and checks the exact dual cardinal polynomial; and
10. checks SHA-256 digests of both deterministic exact solution vectors.

The successful terminal lines include THEOREM: maximum nonnegative H_3 -orbit-constant degree on X_{22} is 69 and CERTIFICATE PASS. Together with Lemma 2, the orbit-constant maximum is also the maximum over arbitrary nonnegative node weights.

Quantity	Exact reported value
Python 3.12.13	CERTIFICATE PASS
Julia 1.11.6	CERTIFICATE PASS
Python source SHA-256	ae615d262ecfc564aed119e5002c4fa9 c4fadf35812002426298caf7c6ab89bf
Julia source SHA-256	5c45650b3baec9889204ac423f8ec62c 9cc0782a03e5c30a8467b9cb01a1efea
Degree-69 solution digest	0865572f76c7b8c9816ce937a79361c8 926abe785ac3f41b06e4f289665bf340
Degree-71 solution digest	c1e620fc9cf6d99623423cf82450540a 5991679e8ded6632bd959914068949f2

Table 2. Two-implementation conformance identities. These hashes identify the currently published bytes; they are not an external archival timestamp.

Minimum safe parsing contract

1. Treat X_{22} , radial projection, orbit ordering, normalized measure, and the real embedding $\sqrt{5} > 0$ as frozen definitions.
2. Treat the three rational free masses as inputs, not fitted decimals.
3. Accept equality only if both rational components of every $\mathbb{Q}(\sqrt{5})$ residual are zero.
4. Accept positivity and the minimum-orbit comparison only through the exact sign method, never binary64 ordering.
5. Distinguish orbit mass from per-node weight.
6. Do not broaden the theorem to movable nodes or signed weights.
7. Use symmetrization only for the nonnegative fixed-node corollary.
8. Do not treat implementation PASS as peer review or independent third-party reproduction.

12. Research directions opened by the result

1. Frequency 23. Construct an exact interior point for the target degree $t_{23} = 71$, then locate the first separating dual.
2. A uniform positivity theorem. Show that the spherical moment vector lies in the relative interior of the convex hull of the Class-I orbit evaluation vectors for all sufficiently large f .
3. Sharp-degree law. Compare the actual maximum degree with the dimension threshold defined by $N(t) \leq \#\{\text{partitions of } f \text{ into at most three parts}\}$.
4. Dual geometry. Study which near-vertex or near-edge orbit is selected by the first negative cardinal polynomial and whether that orbit type obeys a periodic law in f .

5. Condition-aware exact solvers. Use floating linear programming only to propose a point in the positive polytope, then rationalize its free coordinates and certify it over $\mathbb{Q}(\sqrt{5})$. This avoids false infeasibility from ill-conditioned raw moment matrices.

Reproducibility record

Quantity	Exact reported value
Grid	complete radial Class-I X_{22}
Node count	4,842
Symmetry	full icosahedral reflection group H_3
Measure	normalized spherical area, total 1
Exact field	$\mathbb{Q}(\sqrt{5})$, real embedding $\sqrt{5} > 0$
Degree-69 rows/rank	49/49
Degree-69 free orbits	$(9,7,6)$, $(8,8,6)$, $(8,7,7)$
Minimum positive orbit mass	$\approx 0.003874805888537519508839054528148877\dots$ at $(11,11,0)$
Degree-70/71 rows/rank	52/52
Forced negative orbit mass	$\approx -0.006609660200017031631251308016577647\dots$ at $(21,1,0)$
Positive-vector SHA-256	0865572f76c7b8c9816ce937a79361c8 926abe785ac3f41b06e4f289665bf340
Forced-vector SHA-256	c1e620fc9cf6d99623423cf82450540a 5991679e8ded6632bd959914068949f2
Python 3.12.13	CERTIFICATE PASS
Julia 1.11.6	CERTIFICATE PASS
Python source SHA-256	ae615d262ecfc564aed119e5002c4fa9 c4fadf35812002426298caf7c6ab89bf
Julia source SHA-256	5c45650b3baec9889204ac423f8ec62c 9cc0782a03e5c30a8467b9cb01a1efea
Superseded PDF SHA-256	72332ba4f97425b132fe8073aeccfd54 50e19c8f62a3a654969401213dced7a1
Archival fixation	locally hosted only; no DOI

Table 3. Frozen reproducibility record for the corrected v1.1 addendum. Canonical solution digests match the superseded 24 August 2026 calculation.

References

- [1] H. Nozaki and M. Sawa, Note on cubature formulae and designs obtained from group orbits, Canadian Journal of Mathematics 64 (2012), 1359–1377. doi:10.4153/CJM-2011-069-5.
- [2] A. D. McLaren, Optimal numerical integration on a sphere, Mathematics of Computation 17 (1963), 361–383. doi:10.1090/S0025-5718-1963-0159418-2.
- [3] A. S. Popov, The cubature formulas on a sphere that are invariant with respect to the icosahedral group of rotations, Numerical Analysis and Applications 1 (2008), 355–361. doi:10.1134/S199542390804006X.
- [4] A. S. Popov, The search of the best cubature formulas on the sphere that are invariant under the icosahedral rotation group, Numerical Analysis and Applications 16 (2023),

348-358. doi:10.1134/S1995423923040067.

[5] M. A. Newman, Exact Positive Icosahedral Cubature on Fixed Radial Class-I Grids (Paper VI v1.3). doi:10.5281/zenodo.22068873.

[6] Omega Naught public archive, <https://omega-naught.grok.me> and frequency-22 record <https://omega-naught.grok.me/cubature/f22>.

End of corrected exact frequency-22 teaching and verification addendum (v1.1).