

Boundary Reconstruction, Volume-Preserving Transport, and an Invariant Cubic

A self-contained mathematical proof

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Abstract

An explicit twelve-point icosahedral rule reconstructs every harmonic polynomial of degree at most two from its boundary values. We prove that an invertible golden-ratio matrix transports this decoder from a ball to an ellipsoid, together with its elliptic differential operator and boundary measure. The matrix preserves three-dimensional volume and the harmonic cubic $J(x, y, z) = z(x^2 + xy - y^2)$. An explicit orthogonal change of coordinates identifies this invariant with $\sqrt{5}XYZ$. On an aligned icosahedral grid, the invariant is invisible to the twelve samples, and its square obstructs degree-six cubature. One additional, explicitly specified boundary value recovers the coefficient of this cubic. We obtain a complete thirteen-sample decoder on a ten-dimensional field space, prove its compatibility with transport, and show how an exact integer quotient–remainder record preserves finitely encoded sampling data. All reconstruction statements are restricted to their stated field spaces.

1 Definitions and statement of the combined theorem

Work over the real numbers. Put

$$\varphi = \frac{1 + \sqrt{5}}{2}, \quad \varphi' = 1 - \varphi, \quad \tau = \frac{1}{\varphi + \pi}. \quad (1.1)$$

Thus $\varphi + \varphi' = 1$, $\varphi\varphi' = -1$, $\varphi^2 = \varphi + 1$, and $\tau > 0$. Define

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \tau & 0 \\ \tau & \tau & 0 \\ 0 & 0 & -\tau^{-2} \end{pmatrix}, \quad \Psi(\xi) = A\xi. \quad (1.2)$$

For $w = (x, y) \in \mathbb{R}^2$ and $\xi = (x, y, z) \in \mathbb{R}^3$, set

$$\mathcal{N}(w) = x^2 + xy - y^2, \quad J(\xi) = z\mathcal{N}(w). \quad (1.3)$$

Let $\mathbb{B}^3 = \{\xi : \|\xi\| \leq 1\}$, $\mathbb{S}^2 = \partial\mathbb{B}^3$, and let σ be normalized spherical area measure, so that $d\sigma = d\Omega/(4\pi)$ and $\sigma(\mathbb{S}^2) = 1$. For a nonnegative integer m , $\mathcal{P}_{\leq m}$ denotes the real polynomials on $\mathbb{R}^3$ of total degree at most m . Define

$$\mathcal{H}_{\leq 2} = \{u \in \mathcal{P}_{\leq 2} : \Delta u = 0\}. \quad (1.4)$$

Put $r = \sqrt{1 + \varphi^2}$. Let $v_1, \dots, v_{12}$ enumerate

$$\mathcal{V} = \frac{1}{r} \{(0, \pm 1, \pm \varphi), (\pm 1, \pm \varphi, 0), (\pm \varphi, 0, \pm 1)\}, \quad (1.5)$$

where the two signs in each group are independent. Let

$$r_+ = \sqrt{1 + \varphi^2}, \quad r_- = \sqrt{1 + (\varphi')^2}, \quad R = \begin{pmatrix} 1/r_+ & \varphi/r_+ & 0 \\ 1/r_- & \varphi'/r_- & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1.6)$$

The aligned nodes and the additional point are

$$n_i = R^T v_i \quad (1 \leq i \leq 12), \quad p_* = R^T \frac{(1, 1, 1)^T}{\sqrt{3}}. \quad (1.7)$$

Define the ellipsoid, transported nodes, and diffusion matrix by

$$E = AB^3, \quad q_i = An_i, \quad q_* = Ap_*, \quad D = AA^T = \begin{pmatrix} \tau^2 & \tau^2 & 0 \\ \tau^2 & 2\tau^2 & 0 \\ 0 & 0 & \tau^{-4} \end{pmatrix}. \quad (1.8)$$

The transported quadratic space and its cubic extension are

$$\mathcal{H}_A = \{u \circ A^{-1} : u \in \mathcal{H}_{\leq 2}\}, \quad \mathcal{F}_A = \mathcal{H}_A \oplus \mathbb{R}J. \quad (1.9)$$

For $y, q \in \mathbb{R}^3$, write

$$K_A(y, q) = 1 + 3y^T D^{-1}q + \frac{5}{2} \left(3(y^T D^{-1}q)^2 - y^T D^{-1}y \right). \quad (1.10)$$

Theorem 1.1 (Combined reconstruction and transport). *The preceding definitions have the following consequences.*

(i) *The map Ψ is invertible, preserves volume, and satisfies $J \circ \Psi = J$. The space $\mathcal{H}_A$ is the nine-dimensional space of polynomials v of degree at most two satisfying $\nabla \cdot (D\nabla v) = 0$.*

(ii) *For $v \in \mathcal{H}_A$,*

$$v(y) = \frac{1}{12} \sum_{i=1}^{12} v(q_i) K_A(y, q_i) \quad (y \in \mathbb{R}^3). \quad (1.11)$$

(iii) *The space $\mathcal{F}_A$ has dimension ten. Moreover, $J(q_i) = 0$ for every i , whereas*

$$J(q_*) = \frac{\sqrt{5}}{3\sqrt{3}} \neq 0. \quad (1.12)$$

For every $f \in \mathcal{F}_A$, define $\hat{v}(y) = \frac{1}{12} \sum_i f(q_i) K_A(y, q_i)$. Then

$$\boxed{f(y) = \hat{v}(y) + \frac{J(y)}{J(q_*)} (f(q_*) - \hat{v}(q_*))}. \quad (1.13)$$

Thus the thirteen stated boundary measurements uniquely determine f . The twelve aligned measurements alone do not.

(iv) *Under $\nu = A_*\sigma$, the image of normalized spherical measure, the twelve transported nodes have degree-five cubature exactness. No choice of real weights on those twelve nodes has degree-six exactness for ν .*

The following sections prove every assertion of Theorem 1.1 and then establish exact recovery of finitely encoded field records.

2 The spherical sampling rule

Lemma 2.1 (Spherical moments). *For distinct coordinate indices j, k , normalized spherical integration satisfies*

$$\int_{\mathbb{S}} \xi_j^2 d\sigma = \frac{1}{3}, \quad \int_{\mathbb{S}} \xi_j^4 d\sigma = \frac{1}{5}, \quad \int_{\mathbb{S}} \xi_j^2 \xi_k^2 d\sigma = \frac{1}{15}, \quad \int_{\mathbb{S}} x^2 y^2 z^2 d\sigma = \frac{1}{105}. \quad (2.1)$$

Any monomial with an odd exponent in at least one coordinate has integral zero. In particular,

$$\int_{\mathbb{S}} \xi_i \xi_j \xi_k \xi_l d\sigma = \frac{\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}}{15}. \quad (2.2)$$

Proof. Reflection in a coordinate plane changes the sign of an odd monomial and preserves σ , which proves its zero integral. Spherical coordinates give, for every integer $m \geq 0$,

$$\int_{\mathbb{S}} z^{2m} d\sigma = \frac{1}{2} \int_{-1}^1 t^{2m} dt = \frac{1}{2m+1}.$$

Coordinate symmetry gives the same formula for x and y . From $(x^2 + y^2 + z^2)^2 = 1$ on $\mathbb{S}^2$, symmetry yields $1 = 3/5 + 6 \int x^2 y^2 d\sigma$; hence the mixed fourth moment is $1/15$. The tensor formula follows by considering the possible even exponent patterns: four equal indices give $1/5$, two pairs give $1/15$, and all other patterns have an odd exponent.

Next,

$$\frac{1}{5} = \int x^4 (x^2 + y^2 + z^2) d\sigma = \frac{1}{7} + 2 \int x^4 y^2 d\sigma,$$

so $\int x^4 y^2 d\sigma = 1/35$. Expanding $(x^2 + y^2 + z^2)^3 = 1$ now gives

$$1 = \frac{3}{7} + \frac{18}{35} + 6 \int x^2 y^2 z^2 d\sigma.$$

The final moment in (2.1) follows. □

Lemma 2.2 (Degree-five icosahedral cubature). *For every $p \in \mathcal{P}_{\leq 5}$,*

$$\frac{1}{12} \sum_{i=1}^{12} p(v_i) = \int_{\mathbb{S}} p d\sigma. \quad (2.3)$$

The same assertion holds after any fixed orthogonal change of the nodes.

Proof. The node set is invariant under independent coordinate sign changes and cyclic coordinate permutations. Thus its average annihilates every monomial with an odd coordinate exponent. For degree at most five, the remaining monomials have total degree zero, two, or four. Since

$$r^2 = \varphi + 2, \quad r^4 = 5\varphi^2, \quad 1 + \varphi^4 = 3\varphi^2,$$

direct averaging on (1.5) gives

$$\frac{1}{12} \sum_i (v_i)_1^2 = \frac{1 + \varphi^2}{3r^2} = \frac{1}{3}, \quad \frac{1}{12} \sum_i (v_i)_1^4 = \frac{1 + \varphi^4}{3r^4} = \frac{1}{5},$$

and

$$\frac{1}{12} \sum_i (v_i)_1^2 (v_i)_2^2 = \frac{\varphi^2}{3r^4} = \frac{1}{15}.$$

Together with the constant moment, these are all required even moments, up to coordinate permutation. Lemma 2.1 proves equality with spherical integration, and linearity proves (2.3).

If O is orthogonal, then $p \circ O$ has the same degree as p , and σ is orthogonally invariant. Applying (2.3) to $p \circ O$ proves the last assertion. $\square$

Proposition 2.3 (The twelve-sample quadratic decoder). *Let n_i be any orthogonal image of the nodes in (1.5). Every $u \in \mathcal{H}_{\leq 2}$ has the unique form*

$$u(\xi) = a + b^T \xi + \xi^T C \xi, \quad C = C^T, \quad \text{tr } C = 0. \quad (2.4)$$

The space $\mathcal{H}_{\leq 2}$ has dimension nine, and

$$u(\xi) = \frac{1}{12} \sum_{i=1}^{12} u(n_i) \left[1 + 3\xi^T n_i + \frac{5}{2} (3(\xi^T n_i)^2 - \|\xi\|^2) \right] \quad (2.5)$$

for every $\xi \in \mathbb{R}^3$.

Proof. Every quadratic polynomial admits a unique representation with a symmetric quadratic matrix. Its Laplacian is $2 \text{tr } C$, so harmonicity is equivalent to the displayed trace condition. The dimensions are 1, 3, and $6 - 1 = 5$, respectively.

Write $\langle g \rangle = \frac{1}{12} \sum_i g(n_i)$. Products used below have degree at most four, so Lemma 2.2 permits their exact integration. The second and fourth moments in Lemma 2.1 give

$$\langle u \rangle = a, \quad \langle u(n)n \rangle = \frac{b}{3}, \quad \langle u(n)(nn^T - I/3) \rangle = \frac{2C}{15}. \quad (2.6)$$

For the last identity, the constant contribution vanishes, the linear contribution has odd degree, and the quadratic contribution has entries

$$\sum_{k,l} C_{kl} \int n_k n_l n_i n_j \, d\sigma - \frac{\delta_{ij}}{3} \sum_{k,l} C_{kl} \int n_k n_l \, d\sigma = \frac{2C_{ij}}{15},$$

because $C = C^T$ and $\text{tr } C = 0$. Substitution of (2.6) into (2.4) gives (2.5). Its validity for every ξ also proves uniqueness from the twelve samples. $\square$

3 The reversible map and its invariant

Proposition 3.1 (Volume and cubic preservation). *The matrix A is invertible with determinant one. It satisfies*

$$J(A\xi) = J(\xi), \quad A^{-1} = \begin{pmatrix} -\tau^{-1} & \tau^{-1} & 0 \\ \tau^{-1} & 0 & 0 \\ 0 & 0 & -\tau^2 \end{pmatrix}. \quad (3.1)$$

Proof. Since $\det M = -1$, $\det A = \det(\tau M)(-\tau^{-2}) = (-\tau^2)(-\tau^{-2}) = 1$. Multiplication verifies the stated inverse. Linear change of variables therefore shows that Ψ preserves three-dimensional Lebesgue volume. Direct calculation gives

$$\mathcal{N}(M(x, y)) = \mathcal{N}(y, x + y) = y^2 + y(x + y) - (x + y)^2 = -\mathcal{N}(x, y).$$

By homogeneity, $\mathcal{N}(\tau Mw) = -\tau^2 \mathcal{N}(w)$, so

$$J(A(w, z)) = (-\tau^{-2}z)(-\tau^2 \mathcal{N}(w)) = J(w, z).$$

□

4 Transport of the equation, decoder, and measure

Theorem 4.1 (Ellipsoidal reconstruction). *For the matrix A , ellipsoid E , and nodes in (1.8),*

$$E = \{y \in \mathbb{R}^3 : y^T D^{-1} y \leq 1\}, \quad q_i \in \partial E. \quad (4.1)$$

For every $u \in C^2(\mathbb{R}^3)$, with $v = u \circ A^{-1}$,

$$\nabla_y \cdot (D \nabla_y v)(y) = (\Delta u)(A^{-1} y). \quad (4.2)$$

In particular,

$$\mathcal{H}_A = \{v \in \mathcal{P}_{\leq 2} : \nabla \cdot (D \nabla v) = 0\}, \quad (4.3)$$

and $\dim \mathcal{H}_A = 9$. Its reconstruction formula is (1.11).

Proof. The matrix D is symmetric positive definite and $D^{-1} = A^{-T} A^{-1}$. Hence $y^T D^{-1} y = \|A^{-1} y\|^2$, which proves (4.1). The chain rule gives

$$\nabla_y v(y) = A^{-T} \nabla u(A^{-1} y), \quad \text{Hess}_y v(y) = A^{-T} \text{Hess } u(A^{-1} y) A^{-1}.$$

Since D is constant, cyclicity of trace gives

$$\begin{aligned} \nabla_y \cdot (D \nabla_y v)(y) &= \text{tr} \left(A A^T A^{-T} \text{Hess } u(A^{-1} y) A^{-1} \right) \\ &= \text{tr}(\text{Hess } u(A^{-1} y)) = (\Delta u)(A^{-1} y). \end{aligned}$$

The inverse substitution $u = v \circ A$ proves both directions of (4.3). Composition by an invertible linear map preserves polynomial degree and gives a vector-space isomorphism, so the dimension is nine.

For $v = u \circ A^{-1}$, the samples satisfy $v(q_i) = u(n_i)$. Apply Proposition 2.3 at $\xi = A^{-1} y$. As

$$(A^{-1} y)^T n_i = (A^{-1} y)^T (A^{-1} q_i) = y^T D^{-1} q_i, \quad \|A^{-1} y\|^2 = y^T D^{-1} y,$$

the result is precisely (1.11). □

Remark 4.2 (Ordinary harmonicity is not generally transported). The harmonic polynomial $u(x, y, z) = x^2 - y^2$ becomes

$$(u \circ A^{-1})(y_1, y_2, y_3) = \tau^{-2}(y_2^2 - 2y_1 y_2).$$

Its ordinary Laplacian is $2\tau^{-2}$, while its operator $\nabla \cdot (D \nabla \cdot)$ is zero. Thus the operator in Theorem 4.1 is part of the transported specification.

Proposition 4.3 (The transported boundary measure). *Let $\nu = A_*\sigma$, so that $\int_{\partial E} g d\nu = \int_{\mathbb{S}} g(An) d\sigma(n)$. For every $p \in \mathcal{P}_{\leq 5}$,*

$$\frac{1}{12} \sum_{i=1}^{12} p(q_i) = \int_{\partial E} p d\nu. \quad (4.4)$$

If dS_E is Euclidean surface area on ∂E , then at $q = An$,

$$dS_E(q) = \|A^{-T}n\| d\Omega(n), \quad d\nu(q) = \frac{dS_E(q)}{4\pi\|D^{-1}q\|}. \quad (4.5)$$

Proof. The polynomial $p \circ A$ has degree at most five, so Lemma 2.2 proves (4.4). For tangent vectors a, b to the sphere,

$$(Aa) \times (Ab) = \det(A)A^{-T}(a \times b).$$

Taking lengths in a local surface parametrization and using $\det A = 1$ gives the first surface Jacobian. Since $A^{-T}n = A^{-T}A^{-1}q = D^{-1}q$, the definition $d\sigma = d\Omega/(4\pi)$ gives the second identity. The Jacobian is nonconstant: at the first and second coordinate unit vectors its values are $\sqrt{2}\tau^{-1}$ and τ^{-1} , respectively. Thus ν differs from normalized Euclidean surface area on ∂E . $\square$

5 The aligned invariant and the sampling obstruction

Lemma 5.1 (An orthogonal product representation). *The matrix R in (1.6) is orthogonal. With $(X, Y, Z)^T = R(x, y, z)^T$,*

$$J(x, y, z) = \sqrt{5}XYZ. \quad (5.1)$$

Moreover,

$$\Delta J = 0, \quad \nabla \cdot (D\nabla J) = 0, \quad J(n_i) = J(q_i) = 0, \quad J(p_*) = J(q_*) = \frac{\sqrt{5}}{3\sqrt{3}}. \quad (5.2)$$

Proof. The first two rows of R have unit length and their inner product is $(1 + \varphi\varphi')/(r_+r_-) = 0$. The third row is orthogonal to them. Thus $RR^T = I$. Also

$$r_+^2 r_-^2 = (\varphi + 2)(3 - \varphi) = 5, \quad r_+ r_- = \sqrt{5},$$

and

$$(x + \varphi y)(x + \varphi' y) = x^2 + xy - y^2.$$

These equations prove (5.1). Orthogonality also shows that all n_i and p_* lie on $\mathbb{S}^2$.

Direct differentiation gives $\Delta J = 2z - 2z + 0 = 0$. Proposition 3.1 also implies $J \circ A^{-1} = J$, so Theorem 4.1, applied to $u = J$, gives $\nabla \cdot (D\nabla J) = 0$. This use of (4.2) is valid for a cubic because the operator identity holds for every C^2 function.

For $n_i = R^T v_i$, one of the coordinates of $Rn_i = v_i$ is zero. Equation (5.1) therefore gives $J(n_i) = 0$. At p_* , the three transformed coordinates are all $1/\sqrt{3}$, so $J(p_*) = \sqrt{5}/(3\sqrt{3})$. Invariance under A proves the corresponding statements for q_i and q_* . $\square$

Proposition 5.2 (A common reconstruction and cubature obstruction). *The twelve aligned samples cannot determine an arbitrary harmonic polynomial of degree at most three. Furthermore,*

$$\int_{\mathbb{S}} J^2 d\sigma = \int_{\partial E} J^2 d\nu = \frac{1}{21}. \quad (5.3)$$

No real weights on the twelve nodes n_i give degree-six cubature for σ , and no real weights on q_i give degree-six cubature for ν . In both cases degree five is attained by weights $1/12$.

Proof. The nonzero harmonic cubic J and the zero polynomial have the same values on all twelve aligned nodes. This proves the first assertion; the same argument applies to the transported nodes. Orthogonal invariance, Lemma 5.1, and Lemma 2.1 give

$$\int_{\mathbb{S}} J^2 d\sigma = 5 \int_{\mathbb{S}} X^2 Y^2 Z^2 d\sigma = \frac{5}{105} = \frac{1}{21}.$$

Since $J(An) = J(n)$, pushforward of the measure gives the second integral in (5.3). For any real weights a_i , $\sum_i a_i J(n_i)^2 = \sum_i a_i J(q_i)^2 = 0$. As J^2 has degree six and its integral is nonzero, degree-six exactness is impossible. Degree-five exactness follows from Lemma 2.2 and Proposition 4.3. $\square$

Remark 5.3 (Alignment and time sampling). The assertion $J(n_i) = 0$ uses the explicitly aligned nodes (1.7); it is not asserted for an arbitrary orientation of an icosahedron. For these aligned nodes, $J(A^k n_i) = 0$ for every integer k . Thus measurements of this invariant at the transported copies of the same twelve nodes remain zero at every step.

6 The thirteenth measurement and completion of the main proof

Theorem 6.1 (Exact recovery of the invariant coefficient). *Let $f \in \mathcal{F}_A$. Its thirteen samples at $q_1, \dots, q_{12}, q_*$ uniquely determine f by (1.13). One additional measurement with $J \neq 0$ is necessary and sufficient to augment these fixed twelve aligned samples for this field space.*

Proof. Because J is a nonzero cubic, it is not in the quadratic space $\mathcal{H}_A$. Hence the sum in (1.9) is direct and $\dim \mathcal{F}_A = 9 + 1 = 10$. There are unique $v \in \mathcal{H}_A$ and $\beta \in \mathbb{R}$ with $f = v + \beta J$.

By Lemma 5.1, $f(q_i) = v(q_i)$ for $1 \leq i \leq 12$. Theorem 4.1 therefore gives

$$\hat{v}(y) = \frac{1}{12} \sum_i f(q_i) K_A(y, q_i) = v(y).$$

Evaluation at q_* now yields

$$\beta = \frac{f(q_*) - \hat{v}(q_*)}{J(q_*)},$$

whose denominator is the nonzero number in (1.12). Substituting this value into $f = v + \beta J$ proves (1.13).

If two fields have all thirteen values in common, their quadratic parts coincide by the twelve-sample formula, and their cubic coefficients coincide by the last equation. This proves uniqueness. Without an additional measurement, adding any multiple of J leaves all twelve values unchanged. A point with $J \neq 0$ distinguishes these multiples, proving necessity and sufficiency for augmentation of this fixed sampling set. $\square$

Corollary 6.2 (Commutation of transport and reconstruction). *Let $\mathcal{F} = \mathcal{H}_{\leq 2} \oplus \mathbb{R}J$, and define $\mathcal{U}_A f = f \circ A^{-1}$. Let $\mathcal{S}$ evaluate fields at $n_1, \dots, n_{12}, p_*$, and let $\mathcal{S}_A$ evaluate fields at $q_1, \dots, q_{12}, q_*$. Let $\mathcal{D}$ and $\mathcal{D}_A$ be their reconstruction maps, with $\mathcal{D}_A$ given by (1.13). Then*

$$\mathcal{S}_A \mathcal{U}_A = \mathcal{S}, \quad \mathcal{D}_A \mathcal{S}_A = \text{id}_{\mathcal{F}_A}, \quad \mathcal{D}_A \mathcal{S}_A \mathcal{U}_A = \mathcal{U}_A. \quad (6.1)$$

In particular, on admissible sample vectors, $\mathcal{D}_A = \mathcal{U}_A \mathcal{D}$.

Proof. On the original ball, Proposition 2.3 recovers the quadratic part from the twelve aligned values. Evaluation at p_* , where $J(p_*) \neq 0$, then recovers the cubic coefficient by the same subtraction and division as in Theorem 6.1. This defines $\mathcal{D}$ and proves $\mathcal{D}\mathcal{S} = \text{id}_{\mathcal{F}}$.

The identity $J \circ A^{-1} = J$ shows that $\mathcal{U}_A$ maps $\mathcal{F}$ bijectively onto $\mathcal{F}_A$. At every transported sampling point, $(f \circ A^{-1})(Ap) = f(p)$, proving the first identity. The second is Theorem 6.1; their composition gives the third. The same pointwise argument together with uniqueness gives the final assertion on sample vectors arising from these field spaces. $\square$

Proof of Theorem 1.1. Proposition 3.1 proves volume and invariant preservation. Theorem 4.1 proves the characterization, dimension, and decoder of $\mathcal{H}_A$. Lemma 5.1 and Theorem 6.1 prove the ten-dimensional extension and its sampling formula. Proposition 4.3 gives degree-five exactness for ν , and Proposition 5.2 excludes degree six for any real weights. These are all four assertions. $\square$

7 Exact integer records and information accounting

Theorem 7.1 (Quotient–remainder reversibility). *Let Q, S be integers with $0 < Q < S$. For $d \in \mathbb{Z}_{\geq 0}$, define*

$$F(d) = \left\lfloor \frac{Qd}{S} \right\rfloor, \quad \rho(d) = Qd - SF(d). \quad (7.1)$$

The map $\mathcal{L}(d) = (F(d), \rho(d))$ is a bijection from $\mathbb{Z}_{\geq 0}$ onto

$$\mathcal{A}_{Q,S} = \{(a, r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} : 0 \leq r < S, Sa + r \equiv 0 \pmod{Q}\}. \quad (7.2)$$

Its inverse is

$$\mathcal{L}^{-1}(a, r) = \frac{Sa + r}{Q}. \quad (7.3)$$

An ordered finite remainder sequence and the terminal quotient reconstruct every preceding iterate exactly.

Proof. Euclidean division gives $Qd = SF(d) + \rho(d)$ with $0 \leq \rho(d) < S$, so $\mathcal{L}(d) \in \mathcal{A}_{Q,S}$ and (7.3) recovers d . Conversely, for $(a, r) \in \mathcal{A}_{Q,S}$, the formula defines a nonnegative integer d . Then $Qd/S = a + r/S$, whose floor is a ; its remainder is r . This proves both directions of the bijection.

For a trajectory $d_{k+1} = F(d_k)$, the identity $d_k = (Sd_{k+1} + \rho_k)/Q$ reconstructs its preceding term. Backward induction reconstructs the whole finite trajectory from its terminal value and ordered remainder list. Since $F(d) < d$ for positive integers d , every forward trajectory reaches zero in finitely many steps. A proposed ledger is valid only when the reconstructed values are nonnegative integers and all remainder bounds and identities hold. $\square$

Corollary 7.2 (Exact Shannon identity). *If D is a nonnegative integer-valued random variable with finite Shannon entropy, measured in bits, then*

$$H(D) = H(F(D)) + H(\rho(D) \mid F(D)). \quad (7.4)$$

Proof. The bijection in Theorem 7.1 reassigns the probabilities of D to distinct pairs without changing them. Thus $H(D) = H(F(D), \rho(D))$. The discrete entropy chain rule gives (7.4). It can also be obtained directly by writing the pair probability as the product of its first-coordinate marginal and the conditional probability of its second coordinate, and expanding the logarithm in the entropy sum. $\square$

Theorem 7.3 (Composition with finite field records). *Fix the construction parameters and sample labels used in Theorem 6.1. Let $\mathfrak{F} \subseteq \mathcal{F}_A$ be a class whose exact thirteen-sample records admit an integer encoding enc and a decoding function dec with*

$$\text{dec}(\text{enc}(\mathcal{S}_A f)) = \mathcal{S}_A f \quad (f \in \mathfrak{F}). \quad (7.5)$$

Then quotient-remainder storage followed by reversal and field reconstruction recovers every field in $\mathfrak{F}$ exactly:

$$\mathcal{D}_A \circ \text{dec} \circ \mathcal{L}^{-1} \circ \mathcal{L} \circ \text{enc} \circ \mathcal{S}_A = \text{id}_{\mathfrak{F}}. \quad (7.6)$$

Proof. For any $f \in \mathfrak{F}$, first cancel $\mathcal{L}^{-1} \circ \mathcal{L}$ by Theorem 7.1. Next use (7.5) to recover $\mathcal{S}_A f$. Finally, $\mathcal{D}_A \mathcal{S}_A f = f$ by Theorem 6.1. This gives exactly (7.6). $\square$

The finite-encoding hypothesis is explicit. For example, any finite byte record $b = (b_0, \dots, b_{L-1})$, including a length-zero record, has the injective integer encoding

$$\iota(b) = 256^L + \sum_{j=0}^{L-1} b_j 256^{L-1-j}, \quad 0 \leq b_j < 256. \quad (7.7)$$

The base-256 representation consists of a leading digit 1 followed by the L bytes. Its leading digit therefore retains length and any zero bytes. If those bytes specify exact sample values in a declared notation, decoding recovers those values. No assertion is made that arbitrary real sample vectors have finite exact encodings.

8 Scope of the proved result

The reconstructed field belongs to the explicitly stated ten-dimensional space $\mathcal{F}_A$. The cubic mode is identified by the specified additional sample; the twelve aligned values alone leave its coefficient undetermined. The differential operator is transported with the domain, and the cubature measure is $A_*\sigma$. Ordinary Euclidean surface area has the separate Jacobian (4.5). Integer reversal recovers the exact encoded record under (7.5).

Within these hypotheses, the combined identity (7.6) and the commuting relations (6.1) prove compatibility of boundary reconstruction, reversible geometric transport, and exact record recovery. Every sampling formula in this note is a polynomial identity, so its validity on the stated domain follows without a numerical approximation or a discretized PDE solution.