

Localized realization of prescribed incompressible affine motion

A complete proof for the deformation family in OMEGA Paper VIII

19 September 2026

Standalone proof supplement

Related manuscript: *Exact Arithmetic Certification of Geometric Reconstruction and Transport*, Paper VIII of the OMEGA series, by Matthew Alan Newman. DOI: 10.5281/zenodo.22848213. This document supplies a complete derivation; it is not a separately published version of that record.

We prove that a prescribed symmetric, tracefree affine motion on any compact marked set can be realized by a smooth, compactly supported velocity field solving the forced incompressible Navier–Stokes equation. The force vanishes on a fixed open neighborhood of all marked trajectories. The construction starts and ends at rest, works for every positive viscosity, and has uniformly finite kinetic energy. We then specialize the result to Paper VIII, establish its invariant and work identities, and prove the angular-energy formula.

All derivatives are classical. The symbol $|\cdot|$ is the Euclidean norm, $\|\cdot\|_{\text{op}}$ is the matrix operator norm, and $\text{Sym}_0(m)$ denotes the real symmetric $m \times m$ matrices of trace zero. Time ranges over all of $\mathbb{R}$. Compact support below means compact support in space and time.

Theorem 1 (Localized realization). *Let $m \geq 2$, $T > 0$, and $B \in C_c^\infty((0, T); \text{Sym}_0(m))$, extended by zero to all times. Let L solve*

$$\dot{L}(t) = B(t)L(t), \quad L(0) = I_m.$$

For every compact set $\mathcal{K} \subset \mathbb{R}^m$, there exist $u \in C_c^\infty(\mathbb{R}^m \times \mathbb{R}; \mathbb{R}^m)$ and $p \in C_c^\infty(\mathbb{R}^m \times \mathbb{R})$, independent of viscosity, such that, for each $\nu > 0$, there exists $f_\nu \in C_c^\infty(\mathbb{R}^m \times \mathbb{R}; \mathbb{R}^m)$ satisfying

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f_\nu, \quad \text{div } u = 0. \quad (1)$$

The field is zero near and before time 0, and near and after time T . Every marked particle with initial position $X_0 \in \mathcal{K}$ follows exactly $X(t) = L(t)X_0$. There is a bounded open set U containing every such trajectory, for all times, with $f_\nu = 0$ on $U \times \mathbb{R}$. Furthermore,

$$\sup_{t \in \mathbb{R}} \frac{1}{2} \int_{\mathbb{R}^m} |u(X, t)|^2 dX < \infty.$$

Proof. The affine motion. A linear ODE with smooth bounded coefficients has a unique global fundamental matrix L . For completeness, its integral equation $L(t) = I + \int_0^t B(s)L(s)ds$ gives local existence by successive approximations, and the estimate $\|L(t)\|_{\text{op}} \leq \exp(\int_0^t \|B(s)\|_{\text{op}} ds)$ for $t \geq 0$ prevents finite-time escape. For negative time $B = 0$ and $L = I$. The inverse matrix is obtained by solving $\dot{Y} = -YB$, $Y(0) = I$; differentiating YL gives $YL = I$. Jacobi's determinant formula now gives

$$\frac{d}{dt} \det L = (\text{tr } B) \det L = 0, \quad \det L = 1.$$

The matrix L is constant before 0 and after T .

Put $v(X, t) = B(t)X$ and $C(t) = \dot{B}(t) + B(t)^2$. Because B is symmetric, C is symmetric. Define $p_{\text{aff}}(X, t) = -X^T C(t)X/2$. Then

$$\begin{aligned} \text{div } v &= \text{tr } B = 0, & \Delta v &= 0, \\ \partial_t v + (v \cdot \nabla)v &= (\dot{B} + B^2)X = -\nabla p_{\text{aff}}. \end{aligned} \quad (2)$$

Thus (v, p_{aff}) is an unforced affine solution for every viscosity.

One fixed cutoff for the whole motion. The trajectory set

$$\mathcal{T} = \{L(t)X_0 : 0 \leq t \leq T, X_0 \in \mathcal{K}\}$$

is compact, being a continuous image of $[0, T] \times \mathcal{K}$. Choose $0 < R_0 < R_1$ with $\mathcal{T} \subset \{|X| < R_0\}$. An explicit smooth cutoff is obtained from

$$\rho(s) = \begin{cases} e^{-1/s}, & s > 0, \\ 0, & s \leq 0, \end{cases} \quad h(s) = \frac{\rho(s)}{\rho(s) + \rho(1-s)}$$

by setting

$$\chi(X) = 1 - h\left(\frac{|X|^2 - R_0^2}{R_1^2 - R_0^2}\right). \quad (3)$$

The denominator in h is positive for every real s . For every nonnegative integer k , the right derivatives of ρ are sums of powers of $1/s$ times $e^{-1/s}$ and tend to zero as $s \downarrow 0$. Hence ρ and h are smooth, $h = 0$ on $(-\infty, 0]$, and $h = 1$ on $[1, \infty)$. It follows that $\chi \in C_c^\infty(\mathbb{R}^m)$, $0 \leq \chi \leq 1$, $\chi = 1$ on $\{|X| \leq R_0\}$, and $\chi = 0$ on $\{|X| \geq R_1\}$.

A divergence-free extension. Define an antisymmetric tensor and its localized divergence by

$$A_{ij}(X, t) = \frac{v_i X_j - v_j X_i}{m}, \quad u_i(X, t) = \sum_{j=1}^m \partial_j (\chi A_{ij}). \quad (4)$$

Since $A_{ij} = -A_{ji}$ and mixed derivatives commute,

$$\operatorname{div} u = \sum_{i,j} \partial_i \partial_j (\chi A_{ij}) = 0;$$

the terms with $i = j$ vanish and the other terms cancel in pairs. Euler's identity for the degree-one homogeneous vector field v gives

$$\begin{aligned} \sum_j \partial_j (v_i X_j) &= \sum_j X_j \partial_j v_i + m v_i = (m+1)v_i, \\ \sum_j \partial_j (v_j X_i) &= (\operatorname{div} v) X_i + v_i = v_i. \end{aligned}$$

Consequently $\sum_j \partial_j A_{ij} = v_i$. Expanding (4) yields the explicit formula

$$u = \left(\chi + \frac{X \cdot \nabla \chi}{m} \right) BX - \frac{BX \cdot \nabla \chi}{m} X. \quad (5)$$

Therefore $u = BX$ on the open ball $U = \{|X| < R_0\}$, while $u = 0$ outside the closed ball of radius R_1 . In dimension three this is equivalently

$$u = \nabla \times \left(\frac{\chi}{3} (BX \times X) \right),$$

since $\nabla \times (BX \times X) = 3BX - (\operatorname{tr} B)X = 3BX$.

Pressure, force, and support. Set

$$p = \chi p_{\text{aff}}, \quad f_\nu = \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p. \quad (6)$$

These definitions give the momentum equation in (1). On U , the cutoff is locally constant, so the fields and all derivatives in (6) agree with those in (2). Thus $f_\nu = 0$ on $U \times \mathbb{R}$. Outside

the closed outer ball every term vanishes. The force is supported in the closed annulus $R_0 \leq |X| \leq R_1$, during the active time interval.

Choose $0 < a < b < T$ with $\text{supp } B \subset [a, b]$. All derivatives of B vanish outside $[a, b]$; the same is true of the coefficients of u, p, f_ν . Therefore all three fields are smooth and supported in $\{|X| \leq R_1\} \times [a, b]$. This also proves the rest conditions.

The marked trajectories are actual fluid trajectories. Smoothness and the fixed compact space-time support imply that u and $D_X u$ are bounded. The mean value theorem makes u uniformly Lipschitz in space. The particle ODE consequently has unique local solutions, and the uniform bound on u extends them globally. For $X_0 \in \mathcal{K}$ the prescribed curve $X_*(t) = L(t)X_0$ lies in U during $[0, T]$, and at all other times it is at one of its endpoint positions in U . Moreover,

$$\dot{X}_*(t) = B(t)X_*(t) = u(X_*(t), t), \quad X_*(0) = X_0.$$

Uniqueness identifies X_* with the actual particle trajectory. The variational equation for the global flow, together with $\text{div } u = 0$, shows that its Jacobian determinant is one.

Uniform energy bound. Let $B_* = \sup_t \|B(t)\|_{\text{op}}$, $M_\chi = \sup_X |\nabla \chi(X)|$, and let ω_m be the volume of the unit ball in $\mathbb{R}^m$. Formula (5) and $0 \leq \chi \leq 1$ give

$$|u(X, t)| \leq B_* R_1 \left(1 + \frac{2R_1 M_\chi}{m}\right).$$

Since the support lies in the ball of radius R_1 ,

$$\sup_t E(t) \leq \frac{\omega_m}{2} R_1^{m+2} B_*^2 \left(1 + \frac{2R_1 M_\chi}{m}\right)^2 < \infty. \quad (7)$$

The constants depend on the prescribed motion and chosen cutoff, but not on viscosity. All conclusions follow. $\square$

Corollary 2 (The OMEGA deformation family). *Let $m \geq 3$, $T > 0$, and $c, \eta \in C^\infty(\mathbb{R})$, with $c > 0$. Assume $c = 1, \eta = 0$ near and before 0, and that both are constant near and after T . Define*

$$K = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \Phi_t(w, z) = \left(c(t)e^{\eta(t)K}w, c(t)^{-2/(m-2)}z\right).$$

For every compact marked set, Theorem 1 supplies a finite-energy realization with marked trajectories $\Phi_t(X_0)$. The determinant of Φ_t is one. The polynomial

$$J_m(w, z) = (x^2 + xy - y^2) \prod_{j=1}^{m-2} z_j, \quad w = (x, y),$$

is constant along each marked trajectory.

Proof. Put $\gamma = \dot{c}/c$, $\beta = \dot{\eta}$, and

$$B = \text{diag} \left(\gamma I_2 + \beta K, -\frac{2\gamma}{m-2} I_{m-2} \right).$$

Direct multiplication gives $K^T = K$, $K^2 = I$, and $\text{tr } K = 0$. Thus B is symmetric with trace $2\gamma - 2\gamma = 0$. The schedule assumptions make B smooth and compactly supported in $(0, T)$. Differentiating Φ_t gives $\dot{\Phi}_t = B\Phi_t$ and $\Phi_0 = I$, so $\Phi_t = L(t)$ in Theorem 1.

For $H = \begin{pmatrix} 1 & 1/2 \\ 1/2 & -1 \end{pmatrix}$, direct multiplication also gives $K^T H + H K = 0$. Hence

$$\frac{d}{d\eta} (e^{\eta K^T} H e^{\eta K}) = 0, \quad e^{\eta K^T} H e^{\eta K} = H.$$

It follows that $\mathcal{N}(w) = w^T H w = x^2 + xy - y^2$ obeys $\mathcal{N}(ce^{\eta K} w) = c^2 \mathcal{N}(w)$. Also $\det e^{\eta K} = \exp(\eta \operatorname{tr} K) = 1$. The transverse determinant and the product of transverse coordinates both acquire the factor c^{-2} . This cancels the planar factor c^2 in the determinant and in J_m , proving both conclusions. $\square$

The invariant has an exact, limited domain of validity after localization. For the generator in Corollary 2, $BX \cdot \nabla J_m = 0$, while homogeneity gives $X \cdot \nabla J_m = m J_m$. Therefore (5) implies

$$u \cdot \nabla J_m = -J_m (BX \cdot \nabla \chi). \quad (8)$$

This vanishes on the marked trajectories, but can be nonzero in the surrounding cutoff region.

Corollary 3 (Work and viscosity). *The fields of Theorem 1 satisfy*

$$E'(t) + \nu \int_{\mathbb{R}^m} |\nabla u|^2 dX = \int_{\mathbb{R}^m} f_\nu \cdot u dX, \quad E(t) = \frac{1}{2} \int_{\mathbb{R}^m} |u|^2 dX. \quad (9)$$

For nonzero u ,

$$\int_0^T \int_{\mathbb{R}^m} f_\nu \cdot u dX dt = \nu \int_0^T \int_{\mathbb{R}^m} |\nabla u|^2 dX dt > 0.$$

The same nonzero compactly supported velocity cannot satisfy the equation with the same force at two distinct viscosities, even when the pressures are allowed to differ.

Proof. Differentiate the energy under the integral, using the fixed compact support, and dot (1) with u . Integration by parts gives

$$\int u \cdot (u \cdot \nabla) u = 0, \quad \int u \cdot \nabla p = 0, \quad \int u \cdot \Delta u = - \int |\nabla u|^2,$$

where the first two identities use $\operatorname{div} u = 0$. This proves (9). Since $u = 0$ at $t = 0, T$, time integration proves the work equality. If its right-hand side were zero, continuity and nonnegativity would force $\nabla u = 0$ everywhere. Each spatially constant field $u(\cdot, t)$ is then zero by compact support.

If the force and velocity agree at viscosities $\nu_1 \neq \nu_2$, subtract the two momentum equations and integrate their scalar product with u . The pressure difference again contributes zero, leaving $(\nu_1 - \nu_2) \int |\nabla u|^2 = 0$ at each time. The preceding argument forces $u = 0$. $\square$

Proposition 4 (Angular energy for a radial cutoff). *Fix a time and a symmetric tracefree matrix B . Let $\chi(X) = b(r)$, $r = |X|$, be a smooth radial cutoff, and put $s(r) = rb'(r)/m$, $a(r) = b(r) + s(r)$. For probability surface measure σ_{m-1} on S^{m-1} ,*

$$\int_{S^{m-1}} |u(r\omega)|^2 d\sigma_{m-1} = r^2 \operatorname{tr}(B^2) \left\{ \frac{b(r)^2}{m} + \frac{2b(r)s(r) + s(r)^2}{m+2} \right\}. \quad (10)$$

For fixed r , squared speed is a polynomial of degree at most four on the unit sphere. In dimension three, the twelve normalized icosahedral vertices therefore integrate it exactly.

Proof. For $r > 0$, substitute $\nabla\chi = b'(r)\omega$ in (5), and write $q = \omega^T B \omega$. Then

$$u(r\omega) = r\{aB\omega - sq\omega\}, \quad |u|^2 = r^2\{a^2\omega^T B^2\omega + (s^2 - 2as)q^2\}.$$

The identities extend continuously to $r = 0$. Reflection and rotation symmetry give $\int \omega_i \omega_j = \delta_{ij}/m$. To obtain the fourth moments without assuming a cubature formula, write $A = \int \omega_1^4$ and $D = \int \omega_1^2 \omega_2^2$. Rotating $(\omega_1 + \omega_2)/\sqrt{2}$ to the first coordinate and discarding odd terms gives $A = (2A + 6D)/4$, hence $A = 3D$. Squaring $\sum_i \omega_i^2 = 1$ and integrating then gives $mA + m(m-1)D = 1$. Thus

$$A = \frac{3}{m(m+2)}, \quad D = \frac{1}{m(m+2)}.$$

Orthogonally diagonalizing B and expanding q^2 yields

$$\int \omega^T B^2 \omega = \frac{\text{tr}(B^2)}{m}, \quad \int q^2 = \frac{(\text{tr } B)^2 + 2 \text{tr}(B^2)}{m(m+2)}.$$

Use $\text{tr } B = 0$ and $a = b + s$ to obtain (10). Its scalar coefficient can also be written $2b^2/[m(m+2)] + (b+s)^2/(m+2)$, confirming nonnegativity.

For completeness, set $\varphi = (1 + \sqrt{5})/2$. The twelve nodes are all sign choices of

$$\frac{(0, 1, \varphi)}{\sqrt{\varphi+2}}, \quad \frac{(1, \varphi, 0)}{\sqrt{\varphi+2}}, \quad \frac{(\varphi, 0, 1)}{\sqrt{\varphi+2}},$$

with equal weights $1/12$. Coordinate sign symmetries make every monomial with an odd exponent average to zero. Cyclic symmetry and $\varphi^2 = \varphi + 1$ give

$$\mathcal{Q}(1) = 1, \quad \mathcal{Q}(\omega_i^2) = \frac{1}{3}, \quad \mathcal{Q}(\omega_i^4) = \frac{1}{5}, \quad \mathcal{Q}(\omega_i^2 \omega_j^2) = \frac{1}{15} \quad (i \neq j).$$

These are exactly the sphere moments and cover all even monomials of total degree at most five. Together with the odd cases, linearity proves degree-five exactness and hence the asserted angular-energy claim. $\square$

Individual velocity second moments have a stronger requirement. For the displayed node orientation, $\mathcal{Q}(\omega_3^6) = 2/15$, whereas the sphere moment is $1/7$. The latter follows, for example, from the uniform distribution of the third coordinate on $[-1, 1]$ under probability area measure on S^2 . For $B = \text{diag}(1, 1, -2)$ and $z = \omega_3$, $q = 1 - 3z^2$ and

$$u_3/r = -(2a + s)z + 3sz^3.$$

The terms in z^2 and z^4 are integrated exactly, but the z^6 coefficient is $9r^2s^2$. Consequently

$$\mathcal{Q}(u_3^2) - \int_{S^2} u_3^2 d\sigma_2 = 9r^2s^2 \left(\frac{2}{15} - \frac{1}{7} \right) = -\frac{3}{35}r^2s^2.$$

Total kinetic energy also requires radial integration:

$$E = \frac{1}{2}|S^{m-1}| \int_0^\infty r^{m-1} \int_{S^{m-1}} |u(r\omega)|^2 d\sigma_{m-1} dr.$$

Scope and relation to established methods. The force in (6) is selected after the desired motion and cutoff are selected. No prescribed force, fixed control region, or uniform cost over all schedules is asserted. Positive scale is necessary; Corollary 2 excludes $c = 0$. The

finite-energy, support, and exact-trajectory conclusions are analytic; they do not depend on numerical trajectory integration.

The potential in (4) is an instance of the polynomial homotopy mechanism: in the notation of Arnold, Falk, and Winther [2, Theorem 5.1, equation (68)], $(d\kappa + \kappa d)\omega = (r + k)\omega$. A closed form of polynomial degree $r = 1$ and form degree $k = m - 1$ therefore has the factor $1/m$ in its potential. Selecting a field and obtaining its force by applying the equation is the manufactured-solution principle [3, Section 3]. The proof above adds the explicit support and trajectory arguments needed for the stated realization.

Finally, with

$$O = \frac{1}{\sqrt{1 + \varphi^2}} \begin{pmatrix} 1 & -\varphi \\ \varphi & 1 \end{pmatrix}, \quad O^T O = I, \quad O^T K O = \text{diag}(1, -1),$$

the deformation in Corollary 2 becomes, under the fixed orthogonal change of coordinates $\text{diag}(O, I_{m-2})$,

$$\text{diag}(ce^\eta, ce^{-\eta}, c^{-2/(m-2)}, \dots, c^{-2/(m-2)}).$$

This identifies the flow as volume-preserving diagonal strain in rotated coordinates. The construction and the specialization are proved here; priority for their exact combination is not asserted.

References

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