

NEWMAN'S QUANTIZED OMEGA CONTRACTION THEOREM

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ABSTRACT. Let $\phi = (1 + \sqrt{5})/2$, let $\lambda_\Omega = \phi/(\phi + \pi)$, and fix a quantization scale $\tau > 0$. For the floor-quantized recurrence

$$z_0 = x, \quad z_{k+1} = \tau \left\lfloor \frac{\lambda_\Omega z_k}{\tau} \right\rfloor,$$

we prove an exact finite-step decomposition of $\lambda_\Omega^n x$ into the lattice state z_n and a weighted ordered sequence of local remainders. The decomposition gives a uniform residual bound, an explicit inverse formula, and an injective augmented-state representation. Every nonnegative orbit reaches 0 after finitely many steps, while every negative orbit reaches the fixed lattice state $-\tau$. We also give the corresponding affine-coordinate form and record the parameterized extension to arbitrary contractions $0 < \alpha < 1$.

1. INTRODUCTION

Uniform scalar quantization maps a real quantity to a prescribed lattice; a standard survey is given by Gray and Neuhoff [3]. The floor and ceiling identities used below are classical discrete-arithmetic facts [2], and the recurrence is a first-order difference equation in the usual sense [1].

The construction studied here begins with the constants

$$(1) \quad \phi = \frac{1 + \sqrt{5}}{2}, \quad \Omega_0 = \phi + \pi, \quad \lambda_\Omega = \frac{\phi}{\phi + \pi}.$$

For a fixed lattice spacing $\tau > 0$, the continuous contraction by λ_Ω is followed at each step by uniform floor quantization. The discarded fractional part is not ignored: it is retained as an ordered local remainder. The principal result shows that these remainders give an exact weighted decomposition of every continuous iterate and permit exact reconstruction of the initial value.

The special parameter (1) satisfies $0 < \lambda_\Omega < 1/2$. That inequality determines the sign-dependent terminal dynamics: nonnegative lattice states descend to 0, and negative lattice states descend to the unique negative fixed point $-\tau$. Section 3 states and proves the theorem in full. Section 4 gives the affine Omega form. Section 5 records the corresponding parameterized

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extension and identifies the negative fixed points for a general contraction factor.

2. DEFINITIONS AND PRELIMINARY LEMMAS

Fix $\tau > 0$. Define the uniform floor quantizer

$$(2) \quad Q_\tau(y) = \tau \left\lfloor \frac{y}{\tau} \right\rfloor, \quad y \in \mathbb{R}.$$

For an initial value $x \in \mathbb{R}$, define the quantized Omega orbit by

$$(3) \quad z_0 = x, \quad z_{k+1} = Q_\tau(\lambda_\Omega z_k), \quad k \geq 0,$$

and define the local remainder by

$$(4) \quad \varepsilon_k = \lambda_\Omega z_k - z_{k+1}, \quad k \geq 0.$$

For $n \geq 1$, define the accumulated residual

$$(5) \quad r_n(x) = \lambda_\Omega^n x - z_n.$$

Lemma 2.1 (Omega contraction bound). *The parameter λ_Ω satisfies*

$$(6) \quad 0 < \lambda_\Omega < \frac{1}{2}.$$

Proof. Since $1 < \phi < 2$ and $\pi > 3$, one has $\pi > \phi > 0$. Hence

$$\phi + \pi > 2\phi.$$

Both ϕ and $\phi + \pi$ are positive, so division by $\phi + \pi$ gives

$$0 < \frac{\phi}{\phi + \pi} < \frac{1}{2}.$$

This is (6). □

Lemma 2.2 (Quantizer remainder). *For every $y \in \mathbb{R}$,*

$$(7) \quad 0 \leq y - Q_\tau(y) < \tau.$$

Moreover, $Q_\tau(y) \in \tau\mathbb{Z}$.

Proof. The defining property of the floor function gives

$$\left\lfloor \frac{y}{\tau} \right\rfloor \leq \frac{y}{\tau} < \left\lfloor \frac{y}{\tau} \right\rfloor + 1.$$

Multiplication by the positive number τ yields

$$\tau \left\lfloor \frac{y}{\tau} \right\rfloor \leq y < \tau \left\lfloor \frac{y}{\tau} \right\rfloor + \tau.$$

Using (2) and rearranging proves (7). The lattice assertion follows because $\lfloor y/\tau \rfloor \in \mathbb{Z}$. □

3. NEWMAN'S QUANTIZED OMEGA CONTRACTION THEOREM

Theorem 3.1 (Newman's Quantized Omega Contraction Theorem). *Let $x \in \mathbb{R}$, let $\tau > 0$, and define z_k , ε_k , and $r_n(x)$ by (3)–(5). Then, for every integer $n \geq 1$, the following statements hold.*

(i) *The contraction factor satisfies*

$$0 < \lambda_\Omega < \frac{1}{2}.$$

(ii) *For every $k \geq 0$,*

$$(8) \quad \lambda_\Omega z_k = z_{k+1} + \varepsilon_k, \quad 0 \leq \varepsilon_k < \tau.$$

(iii) *The exact finite-step decomposition is*

$$(9) \quad \lambda_\Omega^n x = z_n + \sum_{k=0}^{n-1} \lambda_\Omega^{n-1-k} \varepsilon_k.$$

(iv) *The accumulated residual satisfies*

$$(10) \quad r_n(x) = \sum_{k=0}^{n-1} \lambda_\Omega^{n-1-k} \varepsilon_k$$

and

$$(11) \quad 0 \leq r_n(x) < \tau \frac{1 - \lambda_\Omega^n}{1 - \lambda_\Omega} < \frac{\tau}{1 - \lambda_\Omega} < 2\tau.$$

(v) *One has*

$$(12) \quad z_n \in \tau\mathbb{Z}.$$

(vi) *The augmented state*

$$(13) \quad (z_n, \varepsilon_0, \dots, \varepsilon_{n-1}) \in \tau\mathbb{Z} \times [0, \tau]^n$$

uniquely determines x . Stepwise reconstruction is

$$(14) \quad z_k = \frac{z_{k+1} + \varepsilon_k}{\lambda_\Omega}, \quad k = n-1, n-2, \dots, 0,$$

and the closed reconstruction formula is

$$(15) \quad x = \lambda_\Omega^{-n} z_n + \sum_{k=0}^{n-1} \lambda_\Omega^{-k-1} \varepsilon_k.$$

Consequently, for every fixed $n \geq 1$, the map

$$E_n(x) = (z_n, \varepsilon_0, \dots, \varepsilon_{n-1})$$

is injective.

(vii) *If $x \geq 0$, then the orbit reaches 0 after finitely many iterations. Thus there is an integer $N_+(x) \geq 0$ such that*

$$(16) \quad z_n = 0 \quad (n \geq N_+(x)),$$

and

$$(17) \quad \lim_{n \rightarrow \infty} z_n = 0, \quad \lim_{n \rightarrow \infty} r_n(x) = 0.$$

For $x > 0$, one may take

$$(18) \quad N_+(x) \leq 1 + \left\lfloor \frac{\lambda_\Omega x}{\tau} \right\rfloor,$$

while $N_+(0) = 0$.

(viii) If $x < 0$, then the orbit reaches $-\tau$ after finitely many iterations. Thus there is an integer $N_-(x) \geq 1$ such that

$$(19) \quad z_n = -\tau \quad (n \geq N_-(x)),$$

and

$$(20) \quad \lim_{n \rightarrow \infty} z_n = -\tau, \quad \lim_{n \rightarrow \infty} r_n(x) = \tau.$$

If

$$(21) \quad m_1 = \left\lceil \frac{-\lambda_\Omega x}{\tau} \right\rceil,$$

then one may take $N_-(x) \leq m_1$.

Proof. Part (i) is Lemma 2.1.

For part (ii), equation (4) gives

$$\lambda_\Omega z_k = z_{k+1} + \varepsilon_k.$$

Since $z_{k+1} = Q_\tau(\lambda_\Omega z_k)$, Lemma 2.2, applied to $y = \lambda_\Omega z_k$, gives

$$0 \leq \varepsilon_k < \tau.$$

This proves (8).

For part (iii), we prove (9) by induction on n . When $n = 1$, $z_0 = x$, and (8) with $k = 0$ gives

$$\lambda_\Omega x = z_1 + \varepsilon_0,$$

which is (9). Assume (9) holds for some $n \geq 1$. Multiplying by λ_Ω gives

$$\lambda_\Omega^{n+1} x = \lambda_\Omega z_n + \sum_{k=0}^{n-1} \lambda_\Omega^{n-k} \varepsilon_k.$$

Using $\lambda_\Omega z_n = z_{n+1} + \varepsilon_n$, we obtain

$$\lambda_\Omega^{n+1} x = z_{n+1} + \varepsilon_n + \sum_{k=0}^{n-1} \lambda_\Omega^{n-k} \varepsilon_k = z_{n+1} + \sum_{k=0}^n \lambda_\Omega^{n-k} \varepsilon_k.$$

This is (9) with n replaced by $n + 1$, completing the induction.

For part (iv), subtracting z_n from (9) gives (10). Every summand is nonnegative. Since $0 \leq \varepsilon_k < \tau$,

$$\begin{aligned} r_n(x) &< \tau \sum_{k=0}^{n-1} \lambda_\Omega^{n-1-k} = \tau \sum_{j=0}^{n-1} \lambda_\Omega^j \\ &= \tau \frac{1 - \lambda_\Omega^n}{1 - \lambda_\Omega}. \end{aligned}$$

Because $0 < \lambda_\Omega^n < 1$, this is strictly less than $\tau/(1 - \lambda_\Omega)$. Lemma 2.1 gives $1 - \lambda_\Omega > 1/2$, hence $\tau/(1 - \lambda_\Omega) < 2\tau$. This proves (11).

Part (v) follows directly from (3) and Lemma 2.2: for every $n \geq 1$, $z_n = Q_\tau(\lambda_\Omega z_{n-1}) \in \tau\mathbb{Z}$.

For part (vi), equation (8) and the positivity of λ_Ω give (14). Starting from z_n and ε_{n-1} , this formula uniquely recovers z_{n-1} ; using ε_{n-2} then uniquely recovers z_{n-2} ; continuing backward uniquely recovers $z_0 = x$. Alternatively, division of (9) by λ_Ω^n gives (15) directly. Therefore the augmented state (13) uniquely determines x , and E_n is injective.

For part (vii), assume $x \geq 0$. Since $\lambda_\Omega > 0$ and the floor of a nonnegative number is nonnegative, induction gives $z_k \geq 0$ for every k . For $k \geq 1$, part (v) permits us to write

$$z_k = m_k \tau, \quad m_k \in \mathbb{Z}_{\geq 0}.$$

If $m_k > 0$, then

$$0 \leq z_{k+1} = Q_\tau(\lambda_\Omega z_k) \leq \lambda_\Omega z_k < z_k,$$

so $0 \leq m_{k+1} < m_k$ and therefore $m_{k+1} \leq m_k - 1$. A strictly decreasing sequence of nonnegative integers reaches 0 in finitely many steps. Once $z_n = 0$, equation (3) gives $z_{n+1} = 0$, so (16) follows. Moreover,

$$z_1 = \tau \left\lfloor \frac{\lambda_\Omega x}{\tau} \right\rfloor,$$

so the initial positive lattice coefficient is $m_1 = \lfloor \lambda_\Omega x / \tau \rfloor$; at most m_1 further strict decreases are required, proving (18). For all sufficiently large n , $z_n = 0$, and hence

$$r_n(x) = \lambda_\Omega^n x \longrightarrow 0,$$

which proves (17).

For part (viii), assume $x < 0$. Since $\lambda_\Omega x < 0$,

$$z_1 = \tau \left\lfloor \frac{\lambda_\Omega x}{\tau} \right\rfloor < 0.$$

For $k \geq 1$, write

$$z_k = -m_k \tau, \quad m_k \in \mathbb{Z}_{> 0}.$$

Then

$$\begin{aligned} z_{k+1} &= \tau \left\lfloor \frac{\lambda_\Omega(-m_k \tau)}{\tau} \right\rfloor = \tau \lfloor -\lambda_\Omega m_k \rfloor \\ &= -\tau \lceil \lambda_\Omega m_k \rceil. \end{aligned}$$

Thus

$$(22) \quad m_{k+1} = \lceil \lambda_\Omega m_k \rceil.$$

If $m_k \geq 2$, Lemma 2.1 gives

$$\lambda_\Omega m_k < \frac{m_k}{2} \leq m_k - 1.$$

Since $m_k - 1$ is an integer,

$$m_{k+1} = \lceil \lambda_\Omega m_k \rceil \leq m_k - 1.$$

Also $\lambda_\Omega m_k > 0$, so $m_{k+1} \geq 1$. Hence the positive integer sequence (m_k) decreases by at least one whenever $m_k \geq 2$, and therefore reaches 1 after finitely many steps. When $m_k = 1$,

$$m_{k+1} = \lceil \lambda_\Omega \rceil = 1$$

because $0 < \lambda_\Omega < 1$. Thus 1 is fixed and (19) follows. From

$$m_1 = - \left\lfloor \frac{\lambda_\Omega x}{\tau} \right\rfloor = \left\lceil \frac{-\lambda_\Omega x}{\tau} \right\rceil,$$

there can be at most $m_1 - 1$ strict decreases before the fixed value 1 is reached, so $N_-(x) \leq m_1$. Finally, for all sufficiently large n ,

$$r_n(x) = \lambda_\Omega^n x - z_n = \lambda_\Omega^n x + \tau.$$

Since $0 < \lambda_\Omega < 1$, one has $\lambda_\Omega^n x \rightarrow 0$, which proves (20). $\square$

Remark 3.2 (Scope of the residual estimate). The estimate (11) is a uniform geometric majorant obtained from the separate inequalities $\varepsilon_k < \tau$. No assertion of sharpness is made. The proved asymptotics are $r_n(x) \rightarrow 0$ for $x \geq 0$ and $r_n(x) \rightarrow \tau$ for $x < 0$.

4. AFFINE-COORDINATE FORM

Define the affine Omega contraction

$$(23) \quad \Omega(v) = 1 + \lambda_\Omega(v - 1), \quad v \in \mathbb{R}.$$

Corollary 4.1 (Affine Omega decomposition). *Let $v \in \mathbb{R}$, set $x = v - 1$, and construct z_k and ε_k from (3) and (4). Then, for every $n \geq 1$,*

$$(24) \quad \Omega^n(v) = 1 + \lambda_\Omega^n(v - 1)$$

and

$$(25) \quad \Omega^n(v) = 1 + z_n + \sum_{k=0}^{n-1} \lambda_\Omega^{n-1-k} \varepsilon_k.$$

In particular, $1 + z_n \in 1 + \tau\mathbb{Z}$.

Proof. We prove (24) by induction. For $n = 1$, it is the definition (23). If it holds for n , then

$$\begin{aligned}\Omega^{n+1}(v) &= 1 + \lambda_\Omega(\Omega^n(v) - 1) \\ &= 1 + \lambda_\Omega(\lambda_\Omega^n(v - 1)) = 1 + \lambda_\Omega^{n+1}(v - 1).\end{aligned}$$

Thus (24) holds for all n . Applying (9) to $x = v - 1$ and adding 1 gives (25). The lattice statement follows from (12). $\square$

5. PARAMETERIZED EXTENSION

The proof of Theorem 3.1 separates the exact residual mechanism from the special inequality $\lambda_\Omega < 1/2$. The following proposition records the resulting extension.

Proposition 5.1 (General floor-quantized contraction). *Let $0 < \alpha < 1$, let $\tau > 0$, and let $x \in \mathbb{R}$. Define*

$$(26) \quad u_0 = x, \quad u_{k+1} = Q_\tau(\alpha u_k), \quad \delta_k = \alpha u_k - u_{k+1}.$$

Then, for every $n \geq 1$,

$$(27) \quad \alpha^n x = u_n + \sum_{k=0}^{n-1} \alpha^{n-1-k} \delta_k,$$

where $0 \leq \delta_k < \tau$, and

$$(28) \quad 0 \leq \alpha^n x - u_n < \tau \frac{1 - \alpha^n}{1 - \alpha}.$$

The augmented state $(u_n, \delta_0, \dots, \delta_{n-1})$ uniquely determines x through

$$(29) \quad x = \alpha^{-n} u_n + \sum_{k=0}^{n-1} \alpha^{-k-1} \delta_k.$$

If $x \geq 0$, then u_n reaches 0 in finitely many steps. If $x < 0$, then u_n reaches a fixed state $-m_\infty \tau$, where m_∞ is a positive integer satisfying

$$(30) \quad m_\infty(1 - \alpha) < 1.$$

Conversely, every positive integer m satisfying $m(1 - \alpha) < 1$ gives a fixed state $-m\tau$. Thus the possible negative fixed-state coefficients are precisely

$$(31) \quad m \in \left\{ 1, 2, \dots, \left\lceil \frac{1}{1 - \alpha} \right\rceil - 1 \right\}.$$

In particular, if $0 < \alpha \leq 1/2$, then the unique negative fixed state is $-\tau$.

Proof. Lemma 2.2, applied to $y = \alpha u_k$, gives

$$\alpha u_k = u_{k+1} + \delta_k, \quad 0 \leq \delta_k < \tau.$$

Induction identical to that used for (9) yields (27). Subtracting u_n and summing the geometric series yields (28). Division by α^n gives (29), proving exact reconstruction and injectivity.

If $x \geq 0$, then every $u_k \geq 0$. For $k \geq 1$, write $u_k = q_k \tau$ with $q_k \in \mathbb{Z}_{\geq 0}$. Whenever $q_k > 0$,

$$0 \leq u_{k+1} = Q_\tau(\alpha u_k) \leq \alpha u_k < u_k,$$

so $q_{k+1} < q_k$. Therefore the orbit reaches 0 in finitely many steps.

Now let $x < 0$. For $k \geq 1$, write $u_k = -m_k \tau$ with $m_k \in \mathbb{Z}_{>0}$. Then

$$u_{k+1} = \tau \lfloor -\alpha m_k \rfloor = -\tau \lceil \alpha m_k \rceil,$$

so

$$(32) \quad m_{k+1} = \lceil \alpha m_k \rceil.$$

Because $0 < \alpha < 1$,

$$1 \leq m_{k+1} \leq m_k.$$

Thus (m_k) is a nonincreasing sequence of positive integers and must stabilize. A positive integer m is fixed by (32) exactly when

$$m = \lceil \alpha m \rceil.$$

By the defining property of the ceiling function, this is equivalent to

$$m - 1 < \alpha m \leq m.$$

The upper inequality is automatic because $\alpha < 1$, and the lower inequality is equivalent to (30). Therefore the stabilized coefficient satisfies (30). Conversely, any positive integer satisfying that condition is fixed. The positive integers strictly less than $1/(1 - \alpha)$ are exactly those in (31). If $0 < \alpha \leq 1/2$, then

$$1 < \frac{1}{1 - \alpha} \leq 2,$$

so the set (31) contains only $m = 1$. □

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