

OMEGA Marked Reference Geometry

Constraint Closure of a One-Arcsecond Seed, the Complete Center-Angular Fan,
and Exact Realization in Oriented Euclidean Three-Space

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Abstract

Let

$$\varphi = \frac{1 + \sqrt{5}}{2}, \quad \varepsilon = 1'' = \frac{\pi}{648000},$$

and let $a > 0$. Begin with an oriented Euclidean circle of radius a , center α , and ordered radius endpoints d, b separated by the central angle ε . Extend the chord line outward by a and then by $a\varphi$ on each side. This paper proves that these rules determine more than the visible planar outline: they determine the six marked baseline points, every center-to-mark radius, every pairwise distance, all fifteen pairwise center angles, the corresponding Gram data, and an oriented orthonormal frame for an exact realization in $\mathbb{R}^3$.

Writing $\delta = \varepsilon/2$, $s = \sin \delta$, and $c = \cos \delta$, the six center rays have signed bearings

$$-\theta_2, -\theta_1, -\theta_0, \theta_0, \theta_1, \theta_2,$$

where

$$\theta_0 = \delta, \quad \theta_1 = \frac{\pi}{4} + \frac{\varepsilon}{4}, \quad \theta_2 = \arctan \frac{\varphi^2 + s}{c}.$$

Thus the first extension produces the exact bearing $45^\circ 00' 00.25''$, while the full golden extension produces $69^\circ 05' 41.496848751''$. A single closed dot-product formula gives all fifteen center angles.

In oriented Euclidean three-space the ordered seed generates an orthonormal frame (u, v, n) , and every local mark has the exact form

$$p = \alpha + a(cu + xv),$$

with $x \in \{-\varphi^2 - s, -1 - s, -s, s, 1 + s, \varphi^2 + s\}$. The resulting configuration has intrinsic rank two but is an exact marked subobject of $\mathbb{R}^3$ and may be placed at arbitrary scale and orientation by a rigid motion. The paper also gives a metric transcription of the repeated $a, a\varphi$ outer-scaffold annotations in the original notebook drawings and derives the identities that follow once that outer rule is adopted.

Keywords. marked Euclidean geometry; arcsecond; golden ratio; rigidity; angular closure; Gram matrix; Euclidean three-space; homothety; geometric reconstruction

1 Statement of the construction

The construction begins with four pieces of data:

1. a center α ;

2. a positive scale a ;
3. an oriented Euclidean plane through α ;
4. an ordered pair d, b on the radius- a circle with central separation $\varepsilon = 1''$.

The chord direction is then extended outward by a and by $a\varphi$ on both sides. Once these rules are fixed, the local marked object is rigid. The principal result of this revision is the explicit closure of its metric and angular data and its exact realization in $\mathbb{R}^3$.

Theorem 1.1 (Local constraint-closure theorem). *Fix $a > 0$, $\varepsilon = 1''$, the ordered seed (α, d, b) , and the bilateral extension rule a followed by $a\varphi$. Then the following are uniquely determined up to orientation-preserving Euclidean isometry:*

1. the six marks d_2, d_1, d, b, b_1, b_2 ;
2. all pairwise distances among them and from them to α ;
3. all fifteen pairwise angles subtended at α ;
4. every triangle with vertex α and two baseline marks, including its side lengths, angles, and area;
5. the normalized configuration Gram matrix;
6. an oriented orthonormal frame (u, v, n) in any oriented realization in $\mathbb{R}^3$.

The intrinsic affine span of the local kernel is two-dimensional.

The proof is distributed through Sections 3–7; every item is given by a closed formula.

2 Constants and notation

Let

$$\varphi = \frac{1 + \sqrt{5}}{2}, \quad \varphi^2 = \varphi + 1, \quad (1)$$

and

$$\varepsilon = 1'' = \frac{1}{3600}^\circ = \frac{\pi}{648000}. \quad (2)$$

Set

$$\delta = \frac{\varepsilon}{2}, \quad s = \sin \delta, \quad c = \cos \delta. \quad (3)$$

Numerically,

$$s = 2.424068405545305953 \times 10^{-6}, \quad c = 0.99999999999706194618 \dots$$

The angle ε is dimensionless. Lengths below carry the scale factor a .

3 The one-arcsecond seed

Definition 3.1 (Oriented seed). *Let u be the unit internal bisector directed from α toward the minor arc between d and b . Let v be the positively directed unit chord direction from d toward b . Then*

$$d - \alpha = a(cu - sv), \quad b - \alpha = a(cu + sv). \quad (4)$$

Consequently

$$\|d - \alpha\| = \|b - \alpha\| = a, \quad \angle dab = \varepsilon. \quad (5)$$

The exact chord and sagitta are

$$L_{\text{chord}} = 2as, \quad h_{\text{sag}} = a(1 - c) = 2a \sin^2 \frac{\varepsilon}{4}. \quad (6)$$

The minor-arc length is $a\varepsilon$.

Let

$$m = \frac{d+b}{2} = \alpha + ac u. \quad (7)$$

Thus m is the midpoint of the chord and $\alpha - m = -ac u$.

Proposition 3.2 (Canonical coordinates). *In coordinates with $\alpha = (0, 0)$, $u = (0, -1)$, and $v = (1, 0)$,*

$$d = (-as, -ac), \quad b = (as, -ac). \quad (8)$$

After translating the chord to $y = 0$ instead,

$$d = (-as, 0), \quad b = (as, 0), \quad \alpha = (0, ac). \quad (9)$$

4 Bilateral golden extensions

Define the first and second outward marks by

$$d_1 = d - av, \quad b_1 = b + av, \quad (10)$$

$$d_2 = d - a\varphi^2 v, \quad b_2 = b + a\varphi^2 v. \quad (11)$$

Because $\varphi^2 = 1 + \varphi$,

$$\|d - d_1\| = \|b - b_1\| = a, \quad (12)$$

$$\|d_1 - d_2\| = \|b_1 - b_2\| = a\varphi, \quad (13)$$

and

$$\|d - d_2\| = \|b - b_2\| = a\varphi^2. \quad (14)$$

Introduce the three nonnegative baseline parameters

$$x_0 = s, \quad x_1 = 1 + s, \quad x_2 = \varphi^2 + s. \quad (15)$$

Then all six marks are expressed uniformly as

$$p_{j,\sigma} = \alpha + a(cu + \sigma x_j v), \quad j \in \{0, 1, 2\}, \quad \sigma \in \{-1, +1\}, \quad (16)$$

with

$$(p_{2,-}, p_{1,-}, p_{0,-}, p_{0,+}, p_{1,+}, p_{2,+}) = (d_2, d_1, d, b, b_1, b_2).$$

Corollary 4.1 (Exact total baseline width). *The full endpoint separation is*

$$\|d_2 - b_2\| = 2a(\varphi^2 + s) = 2a\varphi^2 + L_{\text{chord}}. \quad (17)$$

Thus the one-arcsecond chord survives exactly as the excess over the bilateral golden total $2a\varphi^2$.

5 Complete center-angular closure

Equation (16) determines every ray from α to a baseline mark. Define the unsigned bearing of level j from the bisector u by

$$\theta_j = \arctan \frac{x_j}{c}. \quad (18)$$

The signed bearings of $(d_2, d_1, d, b, b_1, b_2)$ are therefore

$$(-\theta_2, -\theta_1, -\theta_0, +\theta_0, +\theta_1, +\theta_2). \quad (19)$$

Theorem 5.1 (Three fundamental bearings). *The three positive bearings are*

$$\theta_0 = \frac{\varepsilon}{2} = 0.5'', \quad (20)$$

$$\theta_1 = \frac{\pi}{4} + \frac{\varepsilon}{4} = 45^\circ 00' 00.25'', \quad (21)$$

$$\theta_2 = \arctan\left(\frac{\varphi^2 + \sin(\varepsilon/2)}{\cos(\varepsilon/2)}\right) \quad (22)$$

$$= 69^\circ 05' 41.496848751 \dots'' . \quad (23)$$

Proof. The first identity is immediate from the seed. For the second,

$$\tan \theta_1 = \frac{1 + \sin \delta}{\cos \delta} = \tan\left(\frac{\pi}{4} + \frac{\delta}{2}\right),$$

and $\delta = \varepsilon/2$. The third follows directly from $x_2 = \varphi^2 + s$. $\square$

The angular increment contributed by the outer golden segment is

$$\gamma := \theta_2 - \theta_1, \quad (24)$$

with the exact tangent identity

$$\tan \gamma = \frac{1}{\sqrt{5}} \tan\left(\frac{\pi}{4} - \frac{\varepsilon}{4}\right). \quad (25)$$

Hence

$$\gamma = 24^\circ 05' 41.246848751 \dots'' . \quad (26)$$

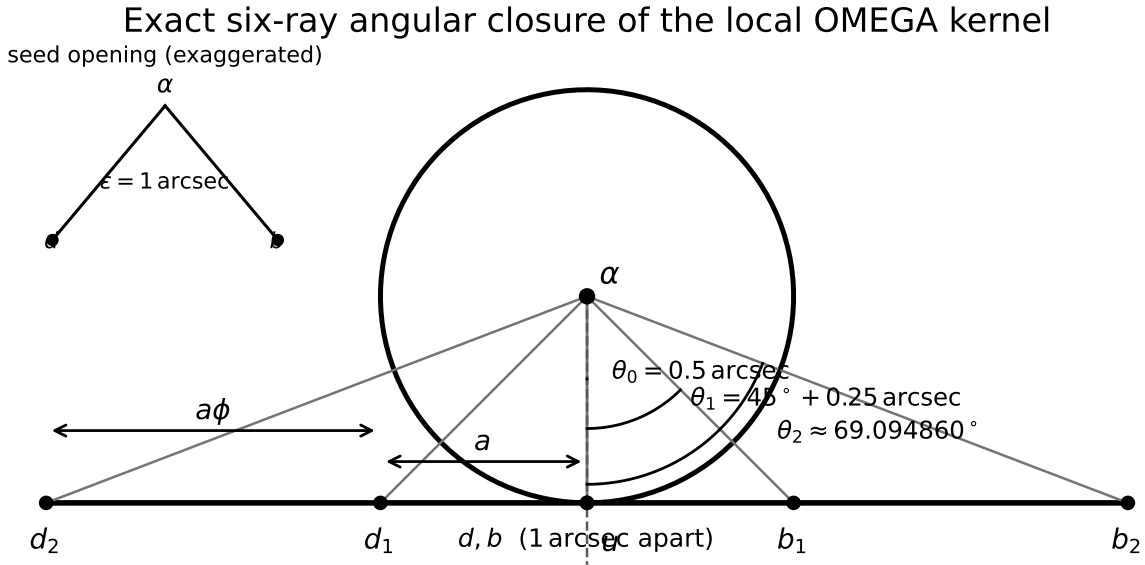


Figure 1: The complete six-ray fan at α . The true d, b separation is one arcsecond and is shown separately in the inset. The other bearings are drawn from their exact values.

5.1 All fifteen center angles

Let β_i denote the signed bearings in (19). Since all six rays lie in a common half-plane and the total span is less than π ,

$$\angle p_i \alpha p_j = |\beta_i - \beta_j|. \quad (27)$$

The numerical angle matrix is shown below; symmetry supplies the lower triangle.

	d_2	d_1	d	b	b_1	b_2
d_2	0	24°05'41.246849''	69°05'40.996849''	69°05'41.996849''	114°05'41.746849''	138°11'22.993698''
d_1		0	44°59'59.750000''	45°00'00.750000''	90°00'00.500000''	114°05'41.746849''
d			0	0°00'01.000000''	45°00'00.750000''	69°05'41.996849''
b				0	44°59'59.750000''	69°05'40.996849''
b_1					0	24°05'41.246849''
b_2						0

The complete outer span is therefore

$$\boxed{\angle d_2 \alpha b_2 = 2\theta_2 = 138^\circ 11' 22.993697502 \dots''}. \quad (28)$$

5.2 One formula for every center angle

For $\sigma, \tau \in \{-1, +1\}$,

$$(p_{j,\sigma} - \alpha) \cdot (p_{k,\tau} - \alpha) = a^2(c^2 + \sigma\tau x_j x_k), \quad (29)$$

and

$$\|p_{j,\sigma} - \alpha\| = a\sqrt{c^2 + x_j^2}. \quad (30)$$

Therefore

$$\boxed{\cos \angle p_{j,\sigma} \alpha p_{k,\tau} = \frac{c^2 + \sigma\tau x_j x_k}{\sqrt{c^2 + x_j^2} \sqrt{c^2 + x_k^2}}}. \quad (31)$$

Equation (31) is the complete center-angular law of the local kernel.

5.3 Exact radial lengths

At the three extension levels,

$$\rho_0 = a, \quad (32)$$

$$\rho_1 = a\sqrt{2(1+s)}, \quad (33)$$

$$\rho_2 = a\varphi\sqrt{3+2s}. \quad (34)$$

The last identity uses $1 + \varphi^4 = 3\varphi^2$. At unit scale,

$$\rho_1 = 1.414215276447263912\dots, \quad (35)$$

$$\rho_2 = 2.802519341384932902\dots \quad (36)$$

5.4 Exact triangle angles

Several triangles expose simple structure that is not visually apparent at one-arcsecond scale.

For $\triangle \alpha d d_1$,

$$\angle d_1 \alpha d = \frac{\pi}{4} - \frac{\varepsilon}{4} = 44^\circ 59' 59.75'', \quad (37)$$

$$\angle \alpha d d_1 = \frac{\pi}{2} + \frac{\varepsilon}{2} = 90^\circ 00' 00.5'', \quad (38)$$

and the remaining angle equals the first. The right-hand triangle $\triangle \alpha b b_1$ is its mirror.

For $\triangle \alpha d_1 d_2$,

$$\angle d_2 \alpha d_1 = \gamma, \quad (39)$$

$$\angle \alpha d_1 d_2 = \frac{3\pi}{4} + \frac{\varepsilon}{4} = 135^\circ 00' 00.25'', \quad (40)$$

$$\angle \alpha d_2 d_1 = \frac{\pi}{2} - \theta_2 = 20^\circ 54' 18.503151249 \dots'' . \quad (41)$$

Again the right side is the mirror image.

More generally, every triangle $\triangle \alpha p_i p_j$ is determined by (16). If z_i, z_j are the signed baseline coordinates from the set

$$Z = (-x_2, -x_1, -x_0, x_0, x_1, x_2), \quad (42)$$

then

$$\|p_i - p_j\| = a|z_i - z_j| \quad (43)$$

and its area is

$$\text{Area}(\alpha, p_i, p_j) = \frac{a^2 c}{2} |z_i - z_j|. \quad (44)$$

6 Gram closure and intrinsic rank

Collect the normalized center-relative landmark vectors as columns in the basis (v, u, n) :

$$X_\Omega = \begin{pmatrix} -x_2 & -x_1 & -x_0 & x_0 & x_1 & x_2 \\ c & c & c & c & c & c \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (45)$$

The physical configuration at scale a is

$$P_\Omega(a) = \alpha \mathbf{1}^T + a X_\Omega. \quad (46)$$

Its normalized Gram matrix is

$$G_\Omega = X_\Omega^T X_\Omega, \quad (G_\Omega)_{ij} = c^2 + z_i z_j. \quad (47)$$

Every center distance and pairwise distance follows from G_Ω by

$$\|p_i - \alpha\|^2 = a^2 (G_\Omega)_{ii}, \quad (48)$$

$$\|p_i - p_j\|^2 = a^2 ((G_\Omega)_{ii} + (G_\Omega)_{jj} - 2(G_\Omega)_{ij}). \quad (49)$$

Theorem 6.1 (Intrinsic rank). *The local OMEGA landmark configuration has*

$$\text{rank } X_\Omega = \text{rank } G_\Omega = 2. \quad (50)$$

Hence its intrinsic Euclidean dimension is two.

Proof. All columns of (45) lie in $\text{span}\{v, u\}$, so the rank is at most two. Since $c \neq 0$ and at least two signed baseline coordinates are distinct, u and v both occur in the column span. Thus the rank is exactly two. The Gram matrix has the same rank as X_Ω . $\square$

7 Exact realization in oriented Euclidean three-space

The rank-two result does not prevent exact use in $\mathbb{R}^3$. It identifies the local kernel as a rigid marked plane carried by an oriented three-dimensional frame.

Let an ordered seed (α, d, b) be given in oriented Euclidean three-space. Put

$$r_d = d - \alpha, \quad r_b = b - \alpha. \quad (51)$$

Define

$$u = \frac{r_d + r_b}{\|r_d + r_b\|}, \quad v = \frac{r_b - r_d}{\|r_b - r_d\|}, \quad n = u \times v. \quad (52)$$

For the one-arcsecond seed these vectors are orthonormal and (u, v, n) is positively oriented.

Theorem 7.1 (Exact 3D reconstruction). *Given α, a , and the ordered one-arcsecond seed in oriented $\mathbb{R}^3$, the entire local OMEGA kernel is reconstructed by*

$$p_{j,\sigma} = \alpha + a(cu + \sigma x_j v), \quad (53)$$

with x_j from (15). Its circular body is

$$C(t) = \alpha + a(\cos t u + \sin t v), \quad \delta \leq t \leq 2\pi - \delta, \quad (54)$$

and its two feet are

$$L_-(\lambda) = d - a\lambda v, \quad (55)$$

$$L_+(\lambda) = b + a\lambda v, \quad (56)$$

for $0 \leq \lambda \leq \varphi^2$. The marks d_1, b_1 occur at $\lambda = 1$ and d_2, b_2 at $\lambda = \varphi^2$.

Thus the exact 3D marked curve is

$$\Omega_a^{(3)} = C([\delta, 2\pi - \delta]) \cup L_-([0, \varphi^2]) \cup L_+([0, \varphi^2]). \quad (57)$$

Every point of the local kernel satisfies

$$(p - \alpha) \cdot n = 0. \quad (58)$$

7.1 Canonical XYZ model

A convenient machine coordinate system is

$$\alpha = (0, 0, 0), \quad u = (0, -1, 0), \quad v = (1, 0, 0), \quad n = (0, 0, 1). \quad (59)$$

Then

$$d_2 = (-a(\varphi^2 + s), -ac, 0), \quad (60)$$

$$d_1 = (-a(1 + s), -ac, 0), \quad (61)$$

$$d = (-as, -ac, 0), \quad (62)$$

$$b = (as, -ac, 0), \quad (63)$$

$$b_1 = (a(1 + s), -ac, 0), \quad (64)$$

$$b_2 = (a(\varphi^2 + s), -ac, 0). \quad (65)$$

Exact OMEGA kernel embedded in oriented Euclidean 3-space

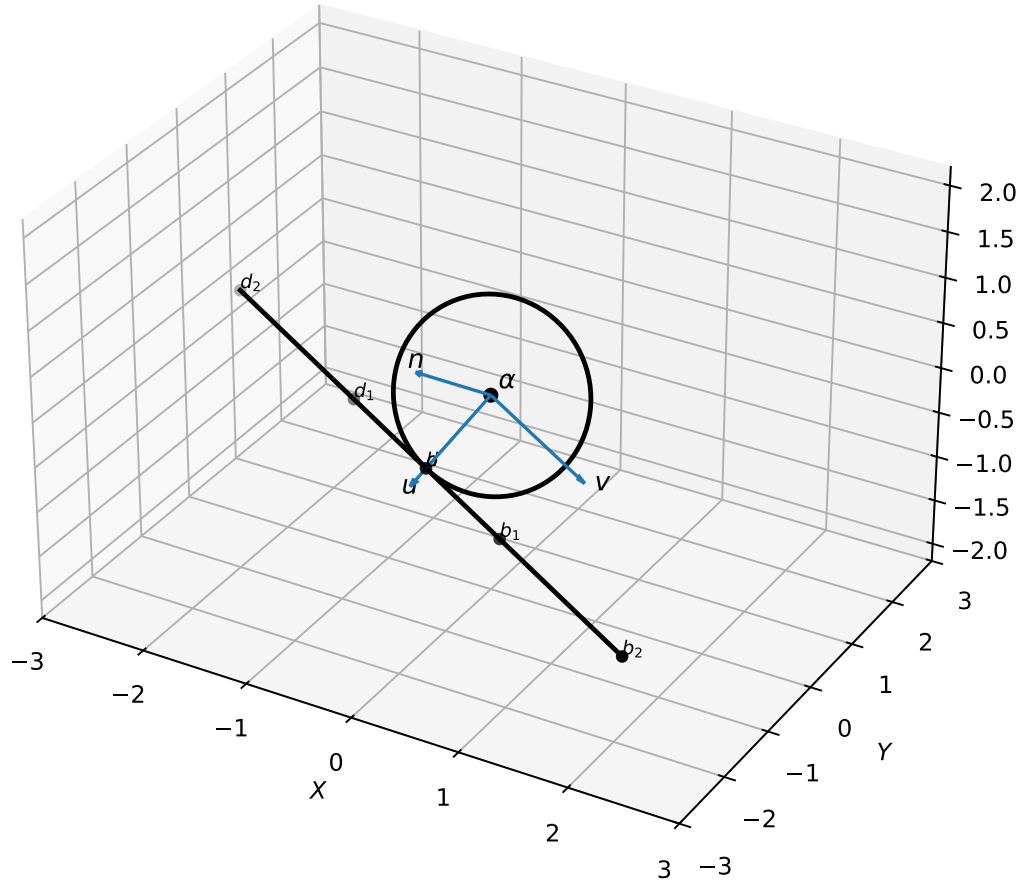


Figure 2: An exact realization of the local OMEGA kernel in oriented Euclidean three-space. The kernel lies in the (u, v) plane; $n = u \times v$ records its oriented normal.

At $a = 1$ these are, to sixteen significant digits,

$$\begin{aligned} d_2 &= (-2.6180364128183005, -0.999999999970619, 0), \\ d_1 &= (-1.0000024240684056, -0.999999999970619, 0), \\ d &= (-0.000002424068405545306, -0.999999999970619, 0), \\ b &= (+0.000002424068405545306, -0.999999999970619, 0), \\ b_1 &= (+1.0000024240684056, -0.999999999970619, 0), \\ b_2 &= (+2.6180364128183005, -0.999999999970619, 0). \end{aligned}$$

7.2 Arbitrary rigid placement

Let $R \in SO(3)$ and $t \in \mathbb{R}^3$. Applying

$$p \mapsto t + Rp \tag{66}$$

places the canonical object anywhere in oriented three-space without changing any distance, center angle, incidence relation, or handedness. Positive scaling by a may be applied before or after the rotation.

This gives a direct construction procedure for CAD, numerical geometry, or visualization: build the canonical coordinates, apply a scale, then apply an element of $SE(3)$.

8 Positive-homothety family

For $c_0 > 0$, define

$$H_{\alpha, c_0}(x) = \alpha + c_0(x - \alpha). \tag{67}$$

Then

$$H_{\alpha, c_0}(K_\Omega(a)) = K_\Omega(c_0 a). \tag{68}$$

Lengths scale by c_0 , squared lengths and Gram entries by c_0^2 , and all angles and normalized Gram data remain unchanged. In particular the complete angular fan of Section 5 is independent of a .

The family $\{K_\Omega(a) : a > 0\}$ therefore carries the free and transitive action

$$(c_0, K_\Omega(a)) \mapsto K_\Omega(c_0 a) \tag{69}$$

of $(\mathbb{R}_{>0}, \times)$.

9 Notebook outer architecture

The original notebook contains a larger construction around the local kernel. Three persistent source marks are labelled *Space*, *Echo*, and *Loco*. We write

$$S = \text{Space}, \quad E = \text{Echo}, \quad L = \text{Loco}. \tag{70}$$

The source drawings are reproduced in Figure 3. They were drawn by the author after the dream in which the configuration first appeared. Their role here is historical and geometric: the exact mathematics is supplied by the explicit rules stated below.

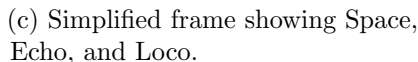
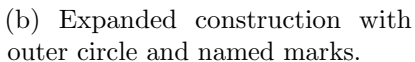
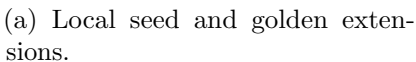


Figure 3: Original notebook source drawings.

9.1 Metric transcription of the repeated $a, a\varphi$ scaffold

The simplified drawing repeats the same two-stage metric pattern used by the local feet. The following definition turns that repeated annotation into an exact outer object.

Let $w = -u$ denote the upward unit normal to the baseline within the kernel plane. Recall that

$$m = \frac{d+b}{2}, \quad \alpha = m + acw. \quad (71)$$

Definition 9.1 (Notebook metric scaffold). *Adopt the repeated $a, a\varphi$ outer annotations as total half-height $a\varphi^2 = a(1 + \varphi)$ above and below the baseline, and use the already determined horizontal endpoints d_2, b_2 as the side coordinates. Define*

$$S = m + a\varphi^2 w, \quad (72)$$

$$E = d_2 - a\varphi^2 w, \quad (73)$$

$$L = b_2 - a\varphi^2 w, \quad (74)$$

and the upper corners

$$T_- = d_2 + a\varphi^2 w, \quad T_+ = b_2 + a\varphi^2 w. \quad (75)$$

Once Definition 9.1 is adopted, the outer scaffold is determined by the same scale a , angle ε , and golden ratio φ .

Proposition 9.2 (Outer frame dimensions). *The scaffold width and height are*

$$W = 2a(\varphi^2 + s), \quad H = 2a\varphi^2, \quad (76)$$

and therefore

$$\boxed{W - H = 2as = L_{\text{chord}}.} \quad (77)$$

Thus the exact departure from a square is the one-arcsecond chord length.

Proposition 9.3 (Center-to-Space distance). *The axial separation from the circle center to Space is*

$$\|S - \alpha\| = a(\varphi^2 - c) = a\varphi + h_{\text{sag}}. \quad (78)$$

Proof. Since $S = m + a\varphi^2 w$ and $\alpha = m + acw$,

$$S - \alpha = a(\varphi^2 - c)w.$$

Using $\varphi^2 = \varphi + 1$ and $h_{\text{sag}} = a(1 - c)$ gives the result. $\square$

Metric outer scaffold obtained by adopting the repeated a , $a\phi$ annotations

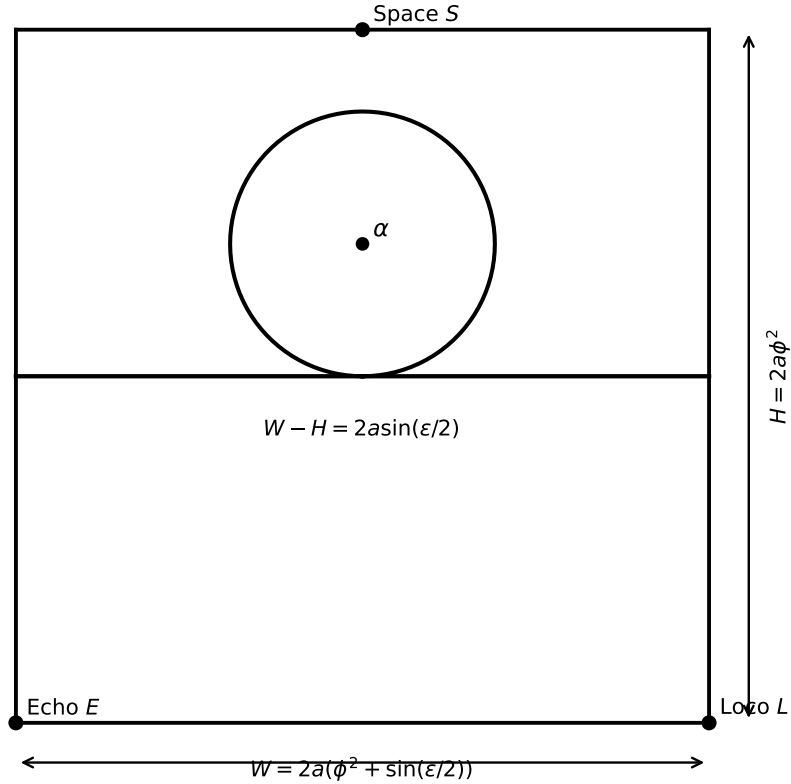


Figure 4: Metric transcription obtained from Definition 9.1. The identity $W - H = L_{\text{chord}}$ follows exactly.

The larger circle and additional curved links visible in the expanded notebook drawing are not required for Theorems 1.1 and 7.1. They can be added to the same coordinate system once their incidences or radii are fixed. The local kernel and the metric scaffold above already provide the anchors against which such circles are solved by ordinary Euclidean equations.

10 From planar rank two to a genuinely nonplanar completion

The local kernel and the metric scaffold above have n -coordinate zero. A genuinely nonplanar point is represented in the same frame by

$$p = \bar{p} + hn, \quad (79)$$

where $\bar{p}$ is its in-plane component and $h \in \mathbb{R}$ is signed depth.

For any in-plane anchor q ,

$$\|p - q\|^2 = \|\bar{p} - q\|^2 + h^2. \quad (80)$$

Hence an independently specified distance $R = \|p - q\|$ gives

$$h = \pm \sqrt{R^2 - \|\bar{p} - q\|^2}. \quad (81)$$

A signed orientation condition such as

$$\text{sgn}((p - \alpha) \cdot n) \quad (82)$$

selects the side of the kernel plane. Multiple independent distance, incidence, or higher-dimensional constraints can be solved simultaneously. Exact three-dimensional completion occurs when the resulting system has a unique solution after the intended rigid-motion freedom is fixed.

This gives a clean hierarchy:

$$\boxed{\text{one-arcsecond seed} \longrightarrow \text{six-ray planar closure} \longrightarrow \text{oriented 3D frame} \longrightarrow \text{signed depth constraints.}} \quad (83)$$

The first three stages are completely explicit in this paper. The last stage is the place for any later nonzero-depth architecture.

11 Machine reconstruction record

For exact reconstruction it is enough to store

$$\mathcal{K} = (\alpha, a, u, v, n, \varepsilon, \varphi). \quad (84)$$

Then compute

$$(s, c) = \left(\sin \frac{\varepsilon}{2}, \cos \frac{\varepsilon}{2} \right), \quad (x_0, x_1, x_2) = (s, 1 + s, \varphi^2 + s), \quad (85)$$

and reconstruct the marks by (53). The circle and feet follow from (54). No individually stored coordinates are required once the frame and scale are known.

A companion reference implementation accompanies this draft. It exports the canonical landmark record as JSON and a sampled OBJ polyline of the exact parametric kernel. The OBJ sampling affects only the displayed circular curve; the landmark coordinates and defining formulas are direct evaluations of the equations above.

12 Conclusion

The local OMEGA object is a closed Euclidean construction. Its input data are the one-arcsecond seed, scale, orientation, and bilateral $a, a\varphi$ extension rule. From these, the complete six-point baseline, radial lengths, pairwise distances, all fifteen center angles, triangle data, and normalized Gram matrix follow without additional geometric choices.

The angular closure has three fundamental positive bearings,

$$\boxed{\theta_0 = \frac{\varepsilon}{2}, \quad \theta_1 = \frac{\pi}{4} + \frac{\varepsilon}{4}, \quad \theta_2 = \arctan \frac{\varphi^2 + \sin(\varepsilon/2)}{\cos(\varepsilon/2)}}. \quad (86)$$

Every center angle is a difference of the corresponding signed bearings, or equivalently an evaluation of (31).

The same data determine an exact oriented realization in $\mathbb{R}^3$ through the frame (u, v, n) and the master point equation

$$\boxed{p_{j,\sigma} = \alpha + a(cu + \sigma x_j v)}. \quad (87)$$

The local kernel has intrinsic rank two; its third basis vector is the oriented normal that permits arbitrary rigid placement in three-space and supplies the axis along which later signed-depth constraints are introduced.

Finally, adopting the repeated outer $a, a\varphi$ annotations as an exact scaffold produces the further identities

$$\boxed{W - H = L_{\text{chord}}} \quad \text{and} \quad \boxed{\|S - \alpha\| = a\varphi + h_{\text{sag}}}. \quad (88)$$

These formulas show how the one-arcsecond chord and sagitta remain present in the larger golden construction.