

OMEGA Marked Reference Geometry

Construction-to-Curve Theorem for a One-Arcsecond Circular Seed
with Bilateral Golden Extensions

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Paper VII in the OMEGA / OMEGA Naught sequence

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Abstract

Let

$$\varphi = \frac{1 + \sqrt{5}}{2}, \quad \varepsilon = 1'' = \frac{\pi}{648000},$$

and let $a > 0$. In an oriented Euclidean plane, choose a center α and an ordered pair d, b on the radius- a circle such that $\angle d\alpha b = \varepsilon$. On the chord line, extend outward from each seed endpoint first by a and then by $a\varphi$. The principal result proves, step by step, that these complete construction data force a unique marked curve: the OMEGA kernel. In particular, the sequential displacement is $a + a\varphi = a\varphi^2$, so the six baseline coordinates are forced to be $\pm s$, $\pm(1 + s)$, and $\pm(\varphi^2 + s)$ in units of a .

This paper gives an existence, uniqueness, and exact-characterization proof for that implication. From the resulting coordinate formula it derives all center-to-mark radii, all pairwise mark distances, the fifteen pairwise center angles, triangle areas, and the normalized Gram matrix. With $\delta = \varepsilon/2$, $s = \sin \delta$, and $c = \cos \delta$, the three positive signed bearing magnitudes are

$$\theta_0 = \delta, \quad \theta_1 = \frac{\pi}{4} + \frac{\varepsilon}{4}, \quad \theta_2 = \arctan\left(\frac{\varphi^2 + s}{c}\right).$$

Every center angle is a difference of the six signed bearings $(-\theta_2, -\theta_1, -\theta_0, \theta_0, \theta_1, \theta_2)$.

The six baseline marks are collinear and have affine dimension one. After the center is included, the configuration has affine dimension two; equivalently, the center-relative landmark matrix and its Gram matrix have rank two. In an oriented realization in $\mathbb{R}^3$, the ordered seed recovers an orthonormal frame (u, v, n) and every mark has the exact form $\alpha + a(cu + \sigma x_j v)$. The planar kernel therefore has an exact embedding at arbitrary position and orientation in $\mathbb{R}^3$, while any genuinely nonplanar completion requires independent depth data. A final section formalizes one explicit metric transcription of the outer notebook scaffold and states separately the identities conditional on that definition.

Keywords. marked Euclidean geometry; arcsecond; golden ratio; distance geometry; Gram matrix; rigid motion; homothety; exact reconstruction

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1 Introduction and scope

The object studied here is a particular marked Euclidean construction. Its ingredients—circle geometry, trigonometric identities, dot products, Gram matrices, rigid motions, and positive homotheties—are

standard. The purpose of the paper is to close the metric specification of the stated construction, not to rename those standard ingredients.

The name *OMEGA* refers to the resulting marked curve, not to an additional axiom. The construction-to-curve theorem below proves the exact implication

$$\left. \begin{array}{l} \text{radius-}a \text{ circle with ordered one-arcsecond seed,} \\ \text{outward steps } a \text{ and } a\varphi \text{ on both sides,} \\ \text{the oriented major circular arc} \end{array} \right\} \implies K_{\Omega}(\alpha, a, u, v). \quad (1)$$

Thus OMEGA is a consequence of the complete recorded construction. The theorem does not assert that the angle $1''$ alone implies the golden extension rule; those are distinct inputs to the construction.

This is Paper VII in a sequence whose earlier works treat quantized floor dynamics, a frozen scale-resolution audit specification, icosahedral sampling and cubature, and the constant $\Omega_0 = \varphi + \pi$ [1, 2, 3, 4, 5, 6]. The results below are self-contained: no theorem from Papers I–VI is needed in their proofs, and SRSD is neither modified nor used as a geometric axiom here. The common sequence name supplies provenance and notation only.

The central precision point is that the one-arcsecond separation does not by itself determine the feet of the OMEGA curve. Exact closure follows from the *combined* data of the one-arcsecond seed, the scale a , the orientation, and the bilateral extension rule a followed by $a\varphi$.

2 Constants, orientation, and notation

Set

$$\varphi = \frac{1 + \sqrt{5}}{2}, \quad \varphi^2 = \varphi + 1, \quad \varepsilon = \frac{\pi}{648000}, \quad (2)$$

and

$$\delta = \frac{\varepsilon}{2}, \quad s = \sin \delta, \quad c = \cos \delta. \quad (3)$$

Thus $0 < \delta \ll 1$, $s > 0$, $c > 0$, and $s^2 + c^2 = 1$. Numerically,

$$s = 2.424068405545306 \times 10^{-6}, \quad c = 0.9999999999970619 \dots$$

Angles are dimensionless; every displayed length retains the scale factor a . The notation $\angle p\alpha q$ denotes the smaller angle between the rays αp and αq .

Unless a dimension is specified, the construction may be realized in an ambient Euclidean space $\mathbb{R}^m$ with $m \geq 2$. Throughout, the Latin letter a is a strictly positive scalar radius, whereas the Greek letter α is the center point. This distinction is part of the notation and should be retained in computational transcription.

Let u be the unit vector from α toward the midpoint of the minor arc from d to b , and let v be the unit chord direction from d to b . The ordered pair (u, v) fixes the orientation of the construction plane.

3 Formal construction and construction-to-OMEGA theorem

Definition 3.1 (Oriented seed). *An oriented one-arcsecond seed at scale a is the tuple (α, a, u, v, d, b) with $a > 0$, $u \cdot v = 0$, $\|u\| = \|v\| = 1$, and*

$$d - \alpha = a(cu - sv), \quad b - \alpha = a(cu + sv). \quad (4)$$

Then $\|d - \alpha\| = \|b - \alpha\| = a$ and $\angle d\alpha b = 2\delta = \varepsilon$.

Definition 3.2 (Bilateral golden extensions). *Starting at each seed endpoint, take an outward segment of length a and then an outward segment of length $a\varphi$. Algebraically,*

$$d_1 = d - av, \quad b_1 = b + av, \quad (5)$$

$$d_2 = d_1 - a\varphi v, \quad b_2 = b_1 + a\varphi v. \quad (6)$$

Since $1 + \varphi = \varphi^2$, the cumulative form is

$$d_2 = d - a\varphi^2 v, \quad b_2 = b + a\varphi^2 v. \quad (7)$$

Introduce the three positive baseline levels

$$x_0 = s, \quad x_1 = 1 + s, \quad x_2 = \varphi^2 + s, \quad (8)$$

and, for $j \in \{0, 1, 2\}$ and $\sigma \in \{-1, +1\}$, set

$$p_{j,\sigma} = \alpha + a(cu + \sigma x_j v). \quad (9)$$

The relevant circle and line parametrizations are

$$C_a(t) = \alpha + a(\cos t u + \sin t v), \quad \delta \leq t \leq 2\pi - \delta, \quad (10)$$

$$L_-(\lambda) = d - a\lambda v, \quad L_+(\lambda) = b + a\lambda v, \quad 0 \leq \lambda \leq \varphi^2. \quad (11)$$

Definition 3.3 (Marked local OMEGA kernel). *The marked local OMEGA kernel is*

$$K_\Omega(\alpha, a, u, v) = C_a([\delta, 2\pi - \delta]) \cup L_-([0, \varphi^2]) \cup L_+([0, \varphi^2]), \quad (12)$$

equipped with center α and ordered marks

$$(p_{2,-}, p_{1,-}, p_{0,-}, p_{0,+}, p_{1,+}, p_{2,+}) = (d_2, d_1, d, b, b_1, b_2). \quad (13)$$

Theorem 3.4 (Construction-to-OMEGA: existence, uniqueness, and exact characterization). *Fix $m \geq 2$. In $\mathbb{R}^m$, the complete data of an oriented seed from [definition 3.1](#), the sequential bilateral rule from [definition 3.2](#), and the choice of the oriented major arc determine exactly one marked curve. That curve is $K_\Omega(\alpha, a, u, v)$ from [definition 3.3](#), and its six marks are exactly the points in (9) with the ordering (13).*

Conversely, $K_\Omega(\alpha, a, u, v)$ satisfies the seed, sequential-extension, and major-arc prescriptions. Hence, for fixed (α, a, u, v) , the complete construction and the displayed kernel are equivalent descriptions of the same marked object. If placement is suppressed, any two realizations with the same scalar data are related by an ambient rigid motion carrying one ordered frame to the other.

Consequently, the following data are determined exactly:

1. the six marks $(d_2, d_1, d, b, b_1, b_2)$;
2. every pairwise distance among α and the six marks;
3. all fifteen pairwise angles subtended by the six marks at α ;
4. every triangle with vertex α and two baseline marks, including its side lengths, angles, and area;
5. the 6×6 matrix of center-relative inner products and its rank;
6. when $m = 3$ and the ambient orientation is fixed, the frame (u, v, n) with $n = u \times v$ and the corresponding exact coordinates.

The six baseline marks are collinear and therefore have affine dimension one. The center-augmented configuration $\{\alpha, d_2, d_1, d, b, b_1, b_2\}$ has affine dimension two. Equivalently, the center-relative landmark matrix and its Gram matrix have rank two.

Proof. Step 1: the seed is forced. By (4),

$$d = \alpha + a(cu - sv) = p_{0,-}, \quad b = \alpha + a(cu + sv) = p_{0,+}.$$

The identities $c^2 + s^2 = 1$ and $c^2 - s^2 = \cos(2\delta)$ give $\|d - \alpha\| = \|b - \alpha\| = a$ and $\angle dab = 2\delta = \varepsilon$.

Step 2: the four extension marks are forced arithmetically. Substituting the seed coordinates into (5) gives

$$d_1 = \alpha + a(cu - (s+1)v) = p_{1,-}, \quad b_1 = \alpha + a(cu + (s+1)v) = p_{1,+}.$$

The second outward step is $a\varphi$. Therefore

$$\begin{aligned} d_2 &= d_1 - a\varphi v = \alpha + a(cu - (s+1+\varphi)v) = \alpha + a(cu - (s+\varphi^2)v) = p_{2,-}, \\ b_2 &= b_1 + a\varphi v = \alpha + a(cu + (s+1+\varphi)v) = \alpha + a(cu + (s+\varphi^2)v) = p_{2,+}, \end{aligned}$$

where the only arithmetic reduction is $1 + \varphi = \varphi^2$. This proves both the master formula (9) and the asserted ordering; no additional mark coordinate is chosen.

Step 3: the circular component is forced. In the oriented plane $\alpha + \text{span}\{u, v\}$, every point on the radius- a circle has the form $C_a(t)$, with t unique modulo 2π . Direct substitution gives

$$C_a(\delta) = b, \quad C_a(2\pi - \delta) = d.$$

Because $0 < \delta < \pi$, the oriented arc from b to d that avoids the interior of the minor seed arc is exactly $C_a([\delta, 2\pi - \delta])$. Thus the prescribed major arc is unique.

Step 4: the linear components are forced. The ray from d opposite the chord direction is $d - \mathbb{R}_{\geq 0}v$, and the ray from b along the chord direction is $b + \mathbb{R}_{\geq 0}v$. Truncating each at the cumulative length $a(1 + \varphi) = a\varphi^2$ gives precisely $L_-([0, \varphi^2])$ and $L_+([0, \varphi^2])$. Their internal marked parameters $\lambda = 1$ and $\lambda = \varphi^2$ are respectively (d_1, b_1) and (d_2, b_2) . Hence their union with the unique major arc is exactly $K_\Omega(\alpha, a, u, v)$. Existence follows from the displayed formulas; uniqueness follows because each component and each marked parameter has been forced.

Step 5: the characterization and metric closure follow. Reading the preceding substitutions in reverse shows that the displayed kernel satisfies every construction prescription, proving the converse. Dot products and coordinate differences then determine all listed distances, angles, areas, and Gram entries. Every mark lies on $\alpha + ac u + \mathbb{R}v$, so the six marks have affine dimension one. Since $c > 0$, α is not on that line; the center-augmented configuration spans u and v and therefore has affine dimension, coordinate rank, and Gram rank two.

Finally, a rigid motion carrying one ordered frame (u, v) to another carries every displayed parametrization to its counterpart. In oriented $\mathbb{R}^3$, $n = u \times v$ completes the frame. This proves uniqueness up to the stated placement freedom and completes the proof. $\square$

4 The one-arcsecond seed

The midpoint of the chord is

$$m = \frac{d+b}{2} = \alpha + ac u. \quad (14)$$

The exact chord length, sagitta, and minor-arc length are

$$L_{\text{chord}} = 2as, \quad h_{\text{sag}} = a(1 - c) = 2a \sin^2\left(\frac{\varepsilon}{4}\right), \quad L_{\text{arc}} = a\varepsilon. \quad (15)$$

Remark 4.1 (Tangent degeneration and the role of $1''$). If the same formulas are extended continuously to $\varepsilon = 0$, then $s = 0$, $c = 1$, and $d = b = \alpha + au$; the baseline becomes tangent to the circle at that common point. The paper studies the specified nonzero instance $\varepsilon = 1''$, for which the contact is split into two ordered points and the minor chord has positive length $2as$. The golden extension rule remains a separate construction datum in either case.

In canonical planar coordinates

$$\alpha = (0,0), \quad u = (0,-1), \quad v = (1,0), \quad (16)$$

one has

$$d = (-as, -ac), \quad b = (as, -ac). \quad (17)$$

After translating the baseline to $y = 0$, the same configuration is

$$d = (-as, 0), \quad b = (as, 0), \quad \alpha = (0, ac). \quad (18)$$

For software using the standard Cartesian $+x$ axis as phase zero, the basis conversion is explicit:

$$(\xi_u, \xi_v) \mapsto (x, y) = (\xi_v, -\xi_u), \quad C_a(t) = (a \sin t, -a \cos t). \quad (19)$$

This is a fixed quarter-turn of coordinates, not a change in the orientation or traversal of the curve.

5 Unit-scale OMEGA curve and golden extensions

The proof of [theorem 3.4](#) supplies the exact unit-scale drawing directly: evaluate (12) at $a = 1$. In the canonical Cartesian coordinates of (16), increasing t is the standard counterclockwise traversal; the displayed interval follows the major arc from b to d .

Corollary 5.1 (Rigid-motion equivariance). *If $F(p) = q + Qp$ is a rigid motion with $Q \in O(m)$, then*

$$F(K_\Omega(\alpha, a, u, v)) = K_\Omega(F(\alpha), a, Qu, Qv).$$

Proof. The major arc and both line segments are the explicit images of the parameter intervals in [equations \(10\)](#) and [\(11\)](#). Substitution of F into those formulas gives the stated equivariance. $\square$

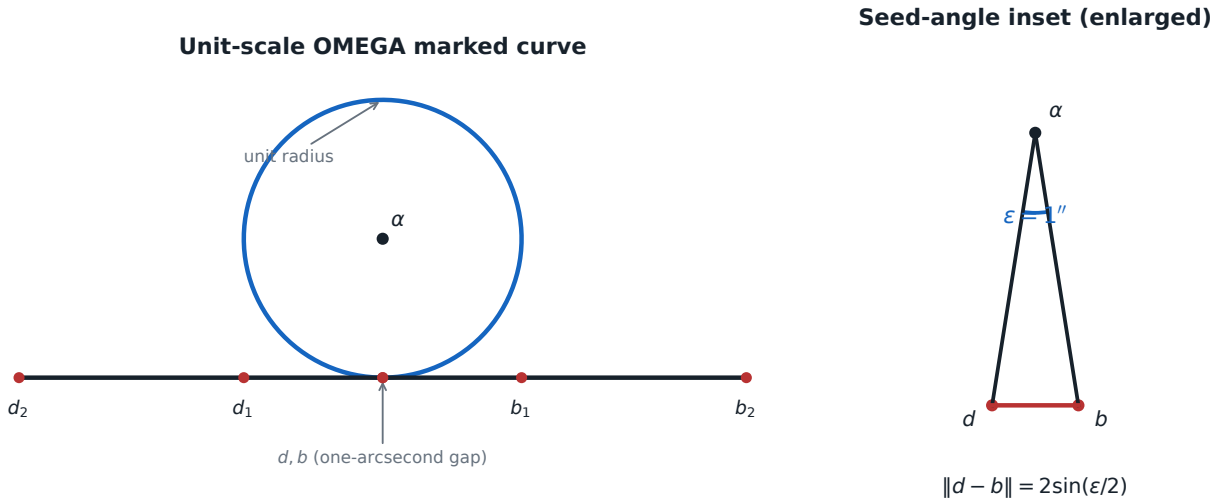


Figure 1: The OMEGA kernel at $a = 1$. The left panel uses the exact unit-scale geometry; the one-arcsecond opening is therefore visually unresolved. The right panel enlarges the seed angle and labels its exact chord length.

The extension lengths are

$$\|d - d_1\| = \|b - b_1\| = a, \quad \|d_1 - d_2\| = \|b_1 - b_2\| = a\varphi. \quad (20)$$

The total baseline width is

$$\|d_2 - b_2\| = 2a(\varphi^2 + s) = 2a\varphi^2 + L_{\text{chord}}. \quad (21)$$

Thus the seed chord is retained exactly as the excess over $2a\varphi^2$.

6 Complete center-angular closure

Define

$$\theta_j = \arctan\left(\frac{x_j}{c}\right), \quad j = 0, 1, 2. \quad (22)$$

All three lie in $(0, \pi/2)$. The signed bearings of $(d_2, d_1, d, b, b_1, b_2)$ relative to u are

$$(-\theta_2, -\theta_1, -\theta_0, \theta_0, \theta_1, \theta_2). \quad (23)$$

Theorem 6.1 (Fundamental bearings). *The positive bearing magnitudes are*

$$\theta_0 = \frac{\varepsilon}{2} = 0.5'', \quad (24)$$

$$\theta_1 = \frac{\pi}{4} + \frac{\varepsilon}{4} = 45^\circ 00' 00.25'', \quad (25)$$

$$\theta_2 = \arctan\left(\frac{\varphi^2 + \sin(\varepsilon/2)}{\cos(\varepsilon/2)}\right) = 69^\circ 05' 41.496848751 \dots''. \quad (26)$$

Proof. Since $s/c = \tan \delta$, the first identity follows. For the second,

$$\frac{1+s}{c} = \tan\left(\frac{\pi}{4} + \frac{\delta}{2}\right),$$

and $\delta = \varepsilon/2$. The third is (22) with $x_2 = \varphi^2 + s$. □

The outer angular increment

$$\gamma = \theta_2 - \theta_1 \quad (27)$$

has the following exact reduction.

Lemma 6.2 (Outer bearing increment).

$$\tan \gamma = \frac{1}{\sqrt{5}} \tan\left(\frac{\pi}{4} - \frac{\varepsilon}{4}\right), \quad (28)$$

and therefore, numerically,

$$\gamma = 24^\circ 05' 41.246848751 \dots''.$$

Proof. Using $\tan \theta_j = x_j/c$ and $x_2 - x_1 = \varphi^2 - 1 = \varphi$,

$$\begin{aligned} \tan(\theta_2 - \theta_1) &= \frac{c(x_2 - x_1)}{c^2 + x_1 x_2} \\ &= \frac{c\varphi}{1 - s^2 + (1+s)(\varphi^2 + s)} \\ &= \frac{c\varphi}{(1 + \varphi^2)(1+s)} = \frac{c}{\sqrt{5}(1+s)}, \end{aligned}$$

where $1 + \varphi^2 = \sqrt{5} \varphi$. Finally, $c/(1+s) = \tan(\pi/4 - \delta/2)$ and $\delta = \varepsilon/2$, which gives (28). □

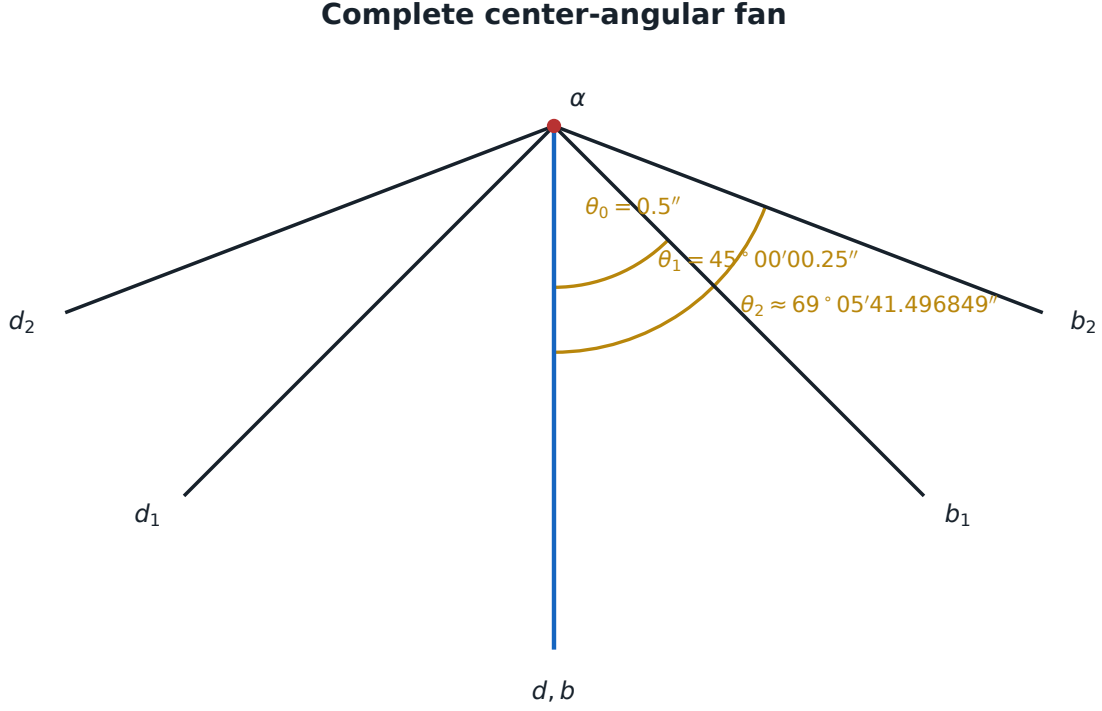


Figure 2: The complete six-ray fan. At the drawing scale the d and b rays coincide visually; their exact separation is $2\theta_0 = 1''$.

Let β_i and β_j be any two signed bearings in (23). Since the full span is $2\theta_2 < \pi$,

$$\boxed{\angle p_i \alpha p_j = |\beta_i - \beta_j|}. \quad (29)$$

This formula gives all fifteen angles. In particular,

$$\angle d_2 \alpha b_2 = 2\theta_2 = 138^\circ 11' 22.993697502 \dots''. \quad (30)$$

The complete pair list appears in [appendix A](#).

7 Metric, triangle, and dot-product closure

Write $z_{j,\sigma} = \sigma x_j$. From (9),

$$(p_{j,\sigma} - \alpha) \cdot (p_{k,\tau} - \alpha) = a^2(c^2 + \sigma\tau x_j x_k), \quad (31)$$

$$\|p_{j,\sigma} - \alpha\| = a\sqrt{c^2 + x_j^2}, \quad (32)$$

$$\|p_{j,\sigma} - p_{k,\tau}\| = a|\sigma x_j - \tau x_k|. \quad (33)$$

Therefore every center angle can also be evaluated directly as

$$\boxed{\cos \angle p_{j,\sigma} \alpha p_{k,\tau} = \frac{c^2 + \sigma\tau x_j x_k}{\sqrt{c^2 + x_j^2} \sqrt{c^2 + x_k^2}}}. \quad (34)$$

At the three extension levels the center radii are

$$\rho_0 = a, \quad (35)$$

$$\rho_1 = a\sqrt{2(1+s)}, \quad (36)$$

$$\rho_2 = a\sqrt{3+2s}. \quad (37)$$

The last formula uses $1 + \varphi^4 = 3\varphi^2$. At $a = 1$,

$$\rho_1 = 1.414215276447264\dots, \quad \rho_2 = 2.802519341384933\dots$$

Every triangle $\triangle \alpha p_i p_j$ has base on the common baseline. If $z_i, z_j \in \{-x_2, -x_1, -x_0, x_0, x_1, x_2\}$ are its signed baseline coordinates, then

$$\boxed{\text{Area}(\alpha, p_i, p_j) = \frac{a^2 c}{2} |z_i - z_j|}. \quad (38)$$

For example, $\triangle \alpha d d_1$ has angles

$$\frac{\pi}{4} - \frac{\varepsilon}{4}, \quad \frac{\pi}{2} + \frac{\varepsilon}{2}, \quad \frac{\pi}{4} - \frac{\varepsilon}{4},$$

and the right-hand triangle $\triangle \alpha b b_1$ is its oriented mirror.

8 Gram closure and affine dimensions

Collect the normalized center-relative landmark vectors as columns in the ordered basis (v, u, n) :

$$z = (-x_2, -x_1, -x_0, x_0, x_1, x_2)^T \in \mathbb{R}^6, \quad \mathbf{1} = (1, 1, 1, 1, 1, 1)^T \in \mathbb{R}^6. \quad (39)$$

$$X_\Omega = \begin{pmatrix} -x_2 & -x_1 & -x_0 & x_0 & x_1 & x_2 \\ c & c & c & c & c & c \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3 \times 6}. \quad (40)$$

At scale a , the physical coordinate array is

$$P_\Omega(a) = \alpha \mathbf{1}^T + a X_\Omega \in \mathbb{R}^{3 \times 6}, \quad (41)$$

where $\alpha \in \mathbb{R}^3$ is written as a column vector. The third row of X_Ω is the coordinate along n and is identically zero. Deleting it gives an intrinsic 2×6 coordinate array with the same Gram matrix. The normalized center-relative Gram matrix is

$$G_\Omega = X_\Omega^T X_\Omega \in \mathbb{R}^{6 \times 6}, \quad (G_\Omega)_{ij} = c^2 + z_i z_j, \quad 1 \leq i, j \leq 6. \quad (42)$$

Equivalently,

$$\boxed{G_\Omega = c^2 \mathbf{1}\mathbf{1}^T + z z^T}. \quad (43)$$

Thus a reproducer need only retain the scalar c and the six-entry vector z ; materializing all 36 Gram entries is optional. The dense matrix is a derived distance-geometry and interoperability object, not the minimal storage form of the construction. It recovers the squared distances by

$$\|p_i - \alpha\|^2 = a^2 (G_\Omega)_{ii}, \quad (44)$$

$$\|p_i - p_j\|^2 = a^2 ((G_\Omega)_{ii} + (G_\Omega)_{jj} - 2(G_\Omega)_{ij}). \quad (45)$$

This is the standard Gram-matrix form of Euclidean distance reconstruction [7].

Proposition 8.1 (Center-relative rank). *The center-relative landmark and Gram matrices satisfy*

$$\text{rank } X_\Omega = \text{rank } G_\Omega = 2. \quad (46)$$

Consequently, the center-augmented configuration has minimal Euclidean embedding dimension two. The six baseline marks alone are collinear and have minimal Euclidean embedding dimension one.

Proof. The row space of (40) is contained in the span of the first two coordinate rows. These rows are independent because $c \neq 0$ and the z_i are not all equal. Hence $\text{rank } X_\Omega = 2$, and $\text{rank}(X_\Omega^T X_\Omega) = \text{rank } X_\Omega$. Centering the augmented configuration at α gives the columns of aX_Ω , so its affine dimension is two. By contrast, subtracting any one baseline mark from the other five produces only scalar multiples of v ; because the marks are distinct, their affine dimension is one. $\square$

Embedding the center-augmented rank-two configuration into $\mathbb{R}^3$, $\mathbb{R}^4$, or a higher-dimensional Euclidean space changes its ambient coordinates but not its center-relative rank, distances, or center angles. The baseline subset remains one-dimensional in every such embedding.

9 Oriented three-space and nonplanar completion

Let the ordered seed be realized in oriented $\mathbb{R}^3$, and set

$$r_d = d - \alpha, \quad r_b = b - \alpha. \quad (47)$$

The frame is recovered directly from the seed:

$$u = \frac{r_d + r_b}{\|r_d + r_b\|}, \quad v = \frac{r_b - r_d}{\|r_b - r_d\|}, \quad n = u \times v. \quad (48)$$

Equal seed radii imply $(r_d + r_b) \cdot (r_b - r_d) = 0$, so (u, v, n) is an orthonormal positively oriented frame.

Proposition 9.1 (Exact realization in $\mathbb{R}^3$). *The marks, circular arc, and feet are reconstructed by equations (9) to (11) in the frame (48). Every point p of the local kernel satisfies*

$$(p - \alpha) \cdot n = 0. \quad (49)$$

For any $R \in SO(3)$ and $t \in \mathbb{R}^3$, the map $p \mapsto t + Rp$ gives an arbitrary orientation-preserving rigid placement without changing any metric or angular datum.

The distinction between embedding and nonplanar completion can be stated exactly.

Proposition 9.2 (Distance-to-anchor determination). *Let $q_0, \dots, q_k \in \mathbb{R}^m$ be known anchors whose affine span A has dimension r , and suppose the distances $R_i = \|p - q_i\|$ are known for an unknown point $p \in \mathbb{R}^m$. If the vectors $q_i - q_0$ span the direction space of A , then the distance data determine the orthogonal projection of p onto A and the magnitude of its component perpendicular to A . They do not, in general, determine the direction of that perpendicular component.*

Proof. Subtract the equation for R_0^2 from each equation for R_i^2 :

$$2p \cdot (q_i - q_0) = \|q_i\|^2 - \|q_0\|^2 - (R_i^2 - R_0^2). \quad (50)$$

Because the differences span the direction space of A , these linear equations determine the projection of p onto A . Substitution into any one distance equation determines the squared norm of the perpendicular component. Rotations within the orthogonal complement leave all anchor distances fixed, so its direction is not determined without further data. $\square$

Canonical placement in oriented Euclidean three-space

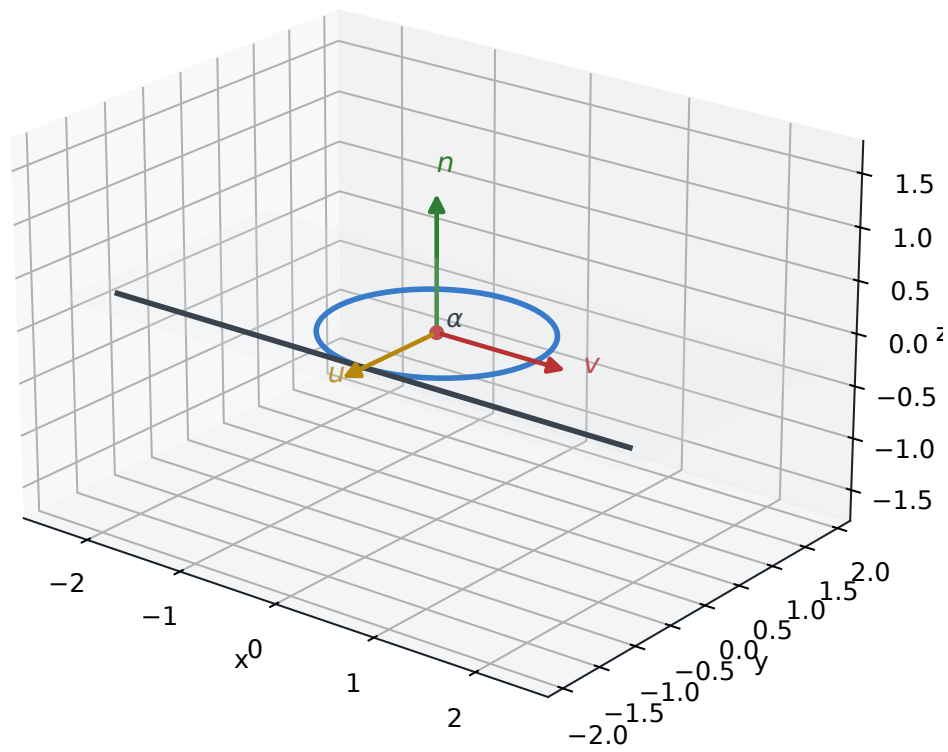


Figure 3: A canonical embedding in oriented $\mathbb{R}^3$. The normal $n = u \times v$ records orientation; the OMEGA kernel itself remains planar and has rank two.

For the OMEGA anchors, $r = 2$. In $\mathbb{R}^3$, writing

$$p = \bar{p} + hn, \quad (51)$$

with $\bar{p}$ in the kernel plane gives

$$\|p - q\|^2 = \|\bar{p} - q\|^2 + h^2 \quad (52)$$

for every in-plane anchor q . The distances leave the mirror ambiguity $h \leftrightarrow -h$; a signed side condition fixes it. In $\mathbb{R}^4$, planar anchor distances leave a circle of perpendicular directions when the depth is nonzero. Independent nonplanar anchors or equivalent incidence/orientation constraints are therefore necessary for a unique genuinely higher-dimensional completion.

10 Positive-homothety family

Fix α, u, v and let

$$\mathcal{F}_{\alpha, u, v} = \{K_{\Omega}(\alpha, a, u, v) : a > 0\}. \quad (53)$$

For $c_0 > 0$, define the center-based homothety

$$H_{\alpha, c_0}(x) = \alpha + c_0(x - \alpha). \quad (54)$$

Then

$$H_{\alpha, c_0}(K_{\Omega}(\alpha, a, u, v)) = K_{\Omega}(\alpha, c_0 a, u, v). \quad (55)$$

Lengths scale by c_0 , areas and Gram entries by c_0^2 , and angles and the normalized Gram matrix remain fixed. Composition and inversion satisfy

$$H_{\alpha, b} \circ H_{\alpha, c} = H_{\alpha, bc}, \quad H_{\alpha, c}^{-1} = H_{\alpha, 1/c}. \quad (56)$$

Thus $(\mathbb{R}_{>0}, \times)$ acts freely and transitively on the scale parameter of the fixed-frame family $\mathcal{F}_{\alpha, u, v}$.

11 Exact transcription of the outer notebook scaffold

The archival drawings contain a larger frame around the local kernel and mark three locations as *Space*, *Echo*, and *Loco*; see [figure 4](#). The local seed does not determine those outer locations until their metric incidences are specified. The next definition makes one precise transcription of the repeated $a, a\varphi$ annotations visible in the frame drawing.

Let $w = -u$ be the upward in-plane unit normal to the baseline. From (14),

$$\alpha = m + acw. \quad (57)$$

Definition 11.1 (Outer metric scaffold). *Adopt total half-height $a\varphi^2 = a(1 + \varphi)$ above and below the baseline and use d_2, b_2 as the side coordinates. Define*

$$S = m + a\varphi^2 w, \quad (58)$$

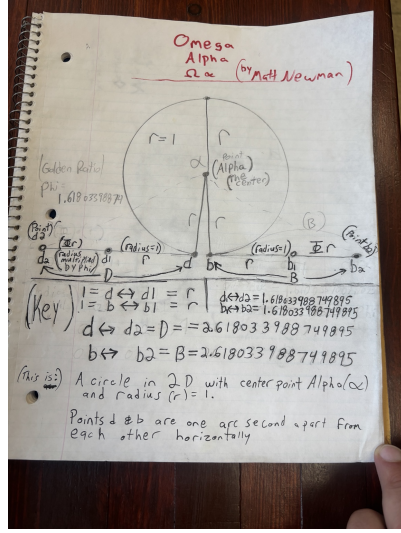
$$E = d_2 - a\varphi^2 w, \quad (59)$$

$$L = b_2 - a\varphi^2 w, \quad (60)$$

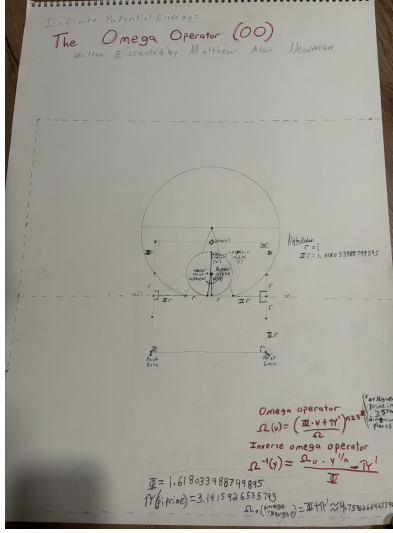
and the upper corners

$$T_- = d_2 + a\varphi^2 w, \quad T_+ = b_2 + a\varphi^2 w. \quad (61)$$

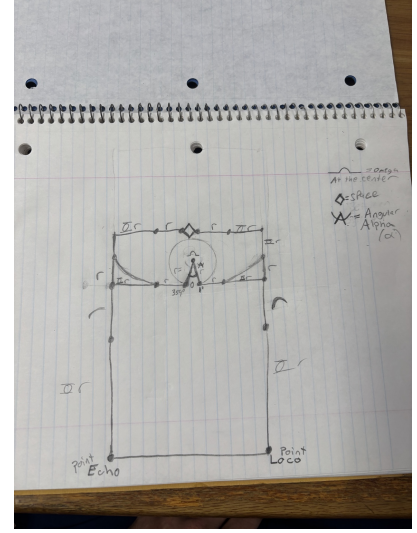
Here S, E, L are the points labelled *Space*, *Echo*, and *Loco*.



(a) Local circle and feet.



(b) Expanded architecture.



(c) Repeated outer scaffold.

Figure 4: Author's archival notebook drawings. The formal geometry used below is the explicit metric definition, not visual measurement from the photographs.

Remark 11.2 (Scope of the scaffold axiom). The equal upper and lower half-heights in [definition 11.1](#) are an explicit metric modeling choice for this paper. They are not asserted to prove the historical intent of the notebook sketch, nor to follow from the circle's lowest tangent. Only consequences conditional on this stated symmetric scaffold are claimed below.

Proposition 11.3 (Outer dimensions). Under [definition 11.1](#), the scaffold width W and height H satisfy

$$W = 2a(\varphi^2 + s), \quad H = 2a\varphi^2, \quad \boxed{W - H = L_{\text{chord}}}. \quad (62)$$

Moreover,

$$\boxed{\|S - \alpha\| = a(\varphi^2 - c) = a\varphi + h_{\text{sag}}}. \quad (63)$$

Proof. The horizontal side coordinates differ by $2a(\varphi^2 + s)$, while the vertical coordinates differ by $2a\varphi^2$; (15) gives $W - H = 2as = L_{\text{chord}}$. Also $S - \alpha = a(\varphi^2 - c)w$, and $\varphi^2 - c = \varphi + (1 - c)$. $\square$

Curved links and any additional circles in the expanded notebook image require their own radii or incidence constraints. They are not used in [theorem 3.4](#) and [propositions 9.1](#) and [11.3](#).

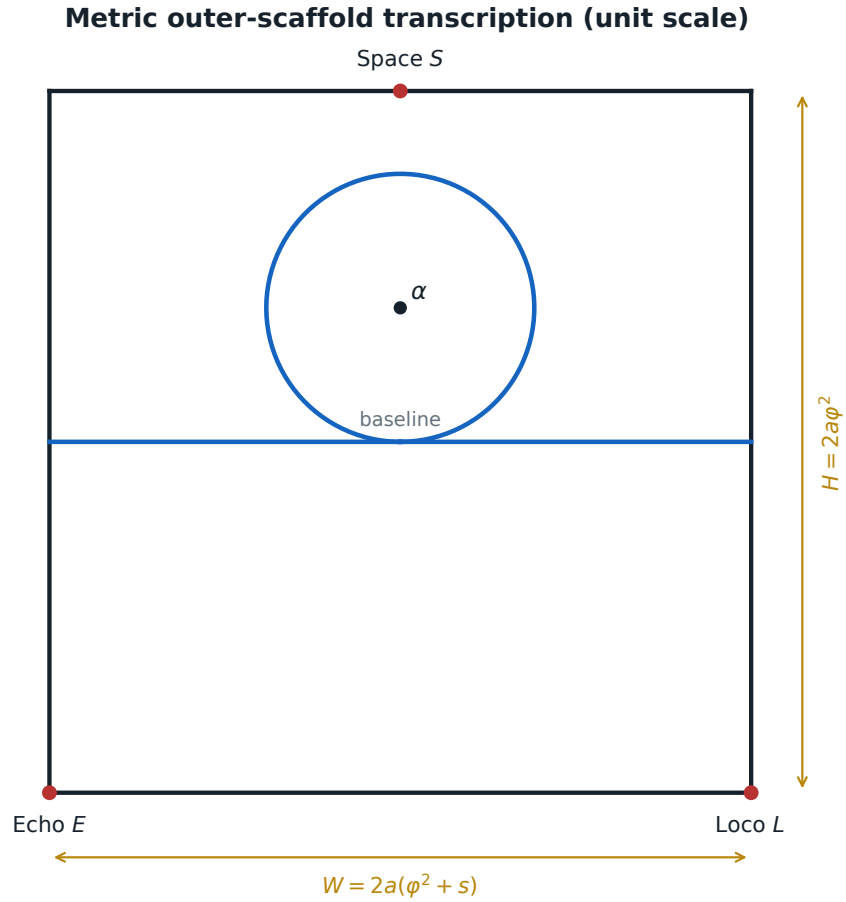


Figure 5: The exact unit-scale scaffold of [definition 11.1](#). Its departure from a square is precisely the one-arcsecond chord length.

12 Computational companion and release semantics

For symbolic reconstruction it is enough to retain

$$\mathcal{K} = (\alpha, a, u, v, \varepsilon, \varphi), \quad (64)$$

with $u \cdot v = 0$ and $\|u\| = \|v\| = 1$. Then compute (s, c) from (3), the x_j from (8), and the marks from (9). In oriented $\mathbb{R}^3$, set $n = u \times v$.

The companion program `omega_reference_geometry.py` exports:

1. a JSON record containing normative symbolic expressions, transform conventions, binary64 coordinate evaluations, the rotated frame, bearings, and the full center-angle matrix; and
2. an OBJ polyline whose six marks are direct evaluations of the defining formulas and whose circular arc is sampled only for display.

The JSON numbers and OBJ vertices are finite-precision approximations, not symbolic exact arithmetic. The exact layer consists of the formulas in this paper and the expression strings carried in the JSON record.

The script `verify_geometry.py` checks the seed radii and angle, extension lengths, bearing identities, Gram rank, pairwise-distance recovery, outer-scaffold identities, rotated-frame normal, parameter validation, and byte agreement with a canonical rebuild of the distributed JSON and OBJ. This is an implementation-conformance check; the proofs are the mathematical justification. The preceding sequence papers and project background are indexed at [8]; the deposited DOI record and this package, not the website, are the source of record for Paper VII.

13 Conclusion

Theorem 3.4 proves the construction-to-OMEGA implication in both directions: the ordered one-arcsecond seed, the bilateral sequential steps $a, a\varphi$, and the major-arc choice produce exactly one marked curve, and the displayed kernel realizes exactly those prescriptions. The cumulative golden step is not an additional fitted coordinate; it is forced arithmetically by $a + a\varphi = a\varphi^2$. Accordingly, OMEGA names the consequence of the complete construction.

Its six marks have the uniform coordinates

$$p_{j,\sigma} = \alpha + a(cu + \sigma x_j v), \quad (x_0, x_1, x_2) = (s, 1 + s, \varphi^2 + s),$$

from which every stated distance, center angle, triangle area, and Gram entry follows. The three positive bearing magnitudes are

$$\theta_0 = \frac{\varepsilon}{2}, \quad \theta_1 = \frac{\pi}{4} + \frac{\varepsilon}{4}, \quad \theta_2 = \arctan\left(\frac{\varphi^2 + \sin(\varepsilon/2)}{\cos(\varepsilon/2)}\right).$$

The six baseline marks have affine dimension one. The center-augmented configuration has affine dimension two, and its center-relative coordinate and Gram matrices have rank two. An ordered seed in oriented $\mathbb{R}^3$ reconstructs the frame (u, v, n) and permits arbitrary rigid placement, but no ambient embedding alone supplies nonzero depth. Unique nonplanar completion requires independent distance, incidence, or signed-orientation data. Under the separately stated outer-scaffold definition, the larger frame also satisfies $W - H = L_{\text{chord}}$ and $\|S - \alpha\| = a\varphi + h_{\text{sag}}$.

A All fifteen center angles

The table lists each unordered pair once. Exact expressions use $\theta_0, \theta_1, \theta_2$ from [theorem 6.1](#) and $\gamma = \theta_2 - \theta_1$.

Pair	Exact angle	Decimal degrees–minutes–seconds
d_2, d_1	γ	$24^\circ 05' 41.246849''$
d_2, d	$\theta_2 - \theta_0$	$69^\circ 05' 40.996849''$
d_2, b	$\theta_2 + \theta_0$	$69^\circ 05' 41.996849''$
d_2, b_1	$\theta_2 + \theta_1$	$114^\circ 05' 41.746849''$
d_2, b_2	$2\theta_2$	$138^\circ 11' 22.993698''$
d_1, d	$\theta_1 - \theta_0$	$44^\circ 59' 59.750000''$
d_1, b	$\theta_1 + \theta_0$	$45^\circ 00' 00.750000''$
d_1, b_1	$2\theta_1$	$90^\circ 00' 00.500000''$
d_1, b_2	$\theta_1 + \theta_2$	$114^\circ 05' 41.746849''$
d, b	$2\theta_0 = \varepsilon$	$0^\circ 00' 01.000000''$
d, b_1	$\theta_0 + \theta_1$	$45^\circ 00' 00.750000''$
d, b_2	$\theta_0 + \theta_2$	$69^\circ 05' 41.996849''$
b, b_1	$\theta_1 - \theta_0$	$44^\circ 59' 59.750000''$
b, b_2	$\theta_2 - \theta_0$	$69^\circ 05' 40.996849''$
b_1, b_2	γ	$24^\circ 05' 41.246849''$

B Canonical unit-scale coordinates

In the XYZ convention

$$\alpha = (0, 0, 0), \quad u = (0, -1, 0), \quad v = (1, 0, 0), \quad n = (0, 0, 1),$$

the exact landmark coordinates at $a = 1$ are

$$\begin{aligned} d_2 &= (-\varphi^2 - s, -c, 0), & d_1 &= (-1 - s, -c, 0), & d &= (-s, -c, 0), \\ b &= (s, -c, 0), & b_1 &= (1 + s, -c, 0), & b_2 &= (\varphi^2 + s, -c, 0). \end{aligned}$$

Their binary64 values in the companion JSON are reproducible from these expressions and are not independent defining data.

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